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Complex Sadiq Emad Eman And Mohand Transforms in Solving Volterra Integro-Differential Equations of the Second Kind and Systems of Volterra Integro Ifferential Equation of the First Kind.

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ABSTRACT

There are different methods and transforms for solving Volterra integro-differential equation of the first kind and the second kind presented by various authors. In this research work wehave studied these various transforms and presented in details the Complex SEE and Mohand transforms for solving Volterra integro-differential equation of the second kind with practical numerical applications. We have also studied in details the Complex SEE and Mohand transforms for solving systems of Volterra integro-differential equation of the first kind and some numerical examples were shown in order to validate the methods.

Keywords: Volterra integro-differential equations, Complex SEE transform, Mohand transform, Convolution, Inverse transforms, Systems of Volterra integro-differential equations.

1. INTRODUCTION

In the early 1900s Volterra introduced some special type of integral equations when he was investigating a population growth model where he focused his study on the hereditary influences. Through this investigation the idea of integro-differential equations was birthed where the equations had both the operators (differential and integral) appearing together in the same equation.Wazwaz A.M (1997)The Volterra integro-differential equationthen appeared in many physical applications such as glass- forming process, Nano hydrodynamics, heat transfer, diffusion process in general, neutron diffusion and biological species coexisting together with increasing and decreasing rates, and wind ripple in the desert. Extensive details of the application of these equations can be found in fields like Biology, Astronomy, Engineering, Physics, Biotechnology, Radiology and many

In recent times integral transformation are one of the most effective and simple mathematical techniques in determining the solutions of advanced problems in many fields like Science, Engineering and Technology Aboodh, K.S. (2014). The major advantage of these transformations is they provide the exact (analytical) solution of the problem without lengthy and ambiguous calculationsAbdelilah, K. and Hassan, S. (2017).

The aim of this research work is to present the methods of solving Volterra Integro- Differential equation of the second kind and systems of Volterra Integro Differential Equations of the first kind.

2. DEFINITION OF COMPLEX SEE AND MOHAND INTEGRAL TRANSFORMS

2.1 Complex SEE Integral Transform

Complex SEE (Sadiq-Emad-Eman) transform denoted by the operator $S^c\{.\}$ defined as

$$S^c\{f(x)\} = F(iv) = \frac{1}{v^n}\int_{x=0}^{\infty} e^{-ivx} f(x)\, dx\, , x \geq 0 \qquad \dots (1)$$

Where i is a complex number and $l_1 \leq v \leq l_2,\ \ n \in \mathbb{Z}$ and $l_1, l_2 > 0$The variable (iv) in this complex convert is used to factor the variable$x$ in the argument of the function $f(x)$.

2.2 Mohand Integral Transform

Mohand transform denoted by the operator M{.} is defined as

$$M\{f(x)\} = F(v) = v^2 \int_{x=0}^{\infty} e^{-vx} f(x)\, dx\, , x \geq 0 \qquad \dots (2)$$

$$l_1 \leq v \leq l_2, \quad n \in \mathbb{Z} \text{ and } l_1, l_2 > 0$$

The variable v in this transform is used to factor the variable x in the argument of the function $f(x)$

2.3 Complex SEE and Mohand Transform of Some Frequently Used Functions.

S/N	**Function**	**Complex SEE Transform**	**Mohand Transform**
1	b	$-\frac{ib}{v^{n+1}}$,Where$b$ is a constant.	v
2	x	$-\frac{1}{v^{n+2}}$	1
3	x^2	$\frac{(2!)i}{v^{n+3}}$	$\frac{(2!)}{v}$
4	x^m	$\frac{(-1)^m(i)^{m-1}(m)!}{v^{n+m+1}} \quad m \in Z^+$	$\frac{(m!)}{v^{m-1}} \quad n \in N$
5	e^{bx}	$-\frac{1}{v^n}\left[\frac{b}{b^2+v^2} + i\frac{v}{b^2+v^2}\right]$	$\frac{v^2}{v-b}$
6	$sin(bx)$	$\frac{-b}{v^n(v^2-b^2)}$	$\frac{bv^2}{v^2+b^2}$
7	$cos(bx)$	$\frac{-iv}{v^n(v^2-b^2)}$	$\frac{v^3}{v^2+b^2}$
8	$sinh(bx)$	$\frac{-b}{v^n(v^2+b^2)}$	$\frac{bv^2}{v^2-b^2}$
9	$cosh(bx)$	$\frac{-iv}{v^n(v^2+b^2)}$	$\frac{v^3}{v^2-b^2}$

2.4 Inverse Complex SEE and Mohand Transform of Some Frequently Used Functions

S/N	**Function (Complex SEE)**	**Function (Mohand)**	**Inverse Function**
1	$S^{c-1}\left\{-\frac{ib}{v^{n+1}}\right\}$	$M^{-1}\{v\}$	$M = 1, S^c = b \text{ where } b \text{ is constant}$
2	$S^{c-1}\left\{-\frac{1}{v^{n+2}}\right\}$	$M^{-1}\{1\}$	x
3	$S^{c-1}\left\{-\frac{(2!)i}{v^{n+3}}\right\}$	$M^{-1}\left\{\frac{1}{v}\right\}$	$S^c = x^2, M = \frac{x^2}{2!}$
4	$S^{c-1}\left\{-\frac{(n!)i}{v^{n+m+1}}\right\}$	$M^{-1}\left\{\frac{1}{v^{m-1}}\right\}, \quad m \in N$	$S^c = x^m, M = \frac{x^2}{n!}$
5	$S^{c-1}\left\{-\frac{1}{v^n}\left[\frac{b}{b^2+v^2} + i\frac{v}{b^2+v^2}\right]\right\}$	$M^{-1}\left\{\frac{v^2}{v-b}\right\}$	e^{bx}
6	$S^{c-1}\left\{\frac{-b}{v^n(v^2-b^2)}\right\}$	$M^{-1}\left\{\frac{bv^2}{v^2+b^2}\right\}$	$sin(bx)$
7	$S^{c-1}\left\{\frac{-iv}{v^n(v^2-b^2)})\right\}$	$M^{-1}\left\{\frac{v^3}{v^2+b^2}\right\}$	$cos(bx)$
8	$S^{c-1}\left\{\frac{-b}{v^n(v^2+b^2)}\right\}$	$M^{-1}\left\{\frac{bv^2}{v^2-b^2}\right\}$	$sinh(bx)$
9	$S^{c-1}\left\{\frac{-iv}{v^n(v^2+b^2)}\right\}$	$M^{-1}\left\{\frac{v^3}{v^2-b^2}\right\}$	$cosh(bx)$

2.5 Fundamental Properties

S/N	Property	Mathematical Form (Complex SEE)	Mathematical Form (Mohand)
1	Linearity	$Scaf1x+bf2x=aSc\{f1x\}+bSc\{f2x\}$	$Maf1x+bf2x=aM\{f1x\}+bM\{f2x\}$
2	Change of Scale	$Sc\ \{f(ax)\} = 1an+1f\ (iva\ )$.	$M\ \{f(ax)\} = af\ (va\ )$.
3	Shifting	$Sc\{eatf(x)\} = (iv-a)nvnF(iv-a)$.	$M\{eatf(x)\} = v2v-aF(v-a)$.
4	1st Derivative	$Scf'x=-f(0)vn+ivFiv$	$Mf'x=vfv-v2f(0)$
5	2nd Derivative	$Scf''x=-f'(0)vn-if0vn-1-v2Fiv$	$Mf''x=v2fv-v3f0-v2f'(0)$
6	nth Derivative	$Scf(m)x=1vn-fm-10-ivfm-20-iv2fm-30-\ldots-ivm-1f0+(iv)mFiv$	$Mf(n)x=$ $vnfv-vn+1f0-vnf'0-\ldots-v2fn-1(0)$
7	Convolution	$Scf1x*f2x=vnF1iv.F2(iv)$	$Mf1x*f2x=1v2M\{f1x\}.M\{f2x\}$

3. SOLVING VOLTERRA INTEGRO-DIFFERENTIAL EQUATION OF THE SECOND KIND.

3.1 Complex SEE Transform

A Volterra integro-differential equation of second kind is defined by:

$$u^{(m)}(x) = f(x) + \lambda \int_0^{(x)} K(x,t)\,u(t)dt \qquad \ldots(3)$$

With

$$u(0) = \delta_0,\ \ u'(0) = \delta_1,\ u''(0) = \delta_2, \ldots\ \ , u^{(m-1)}(0) = \delta_{m-1} \qquad \ldots(4)$$

Where$K(x-t)$ are kernel of the equation.

$u(x)$is the unknown function of x,

$u^{(m)}(x)$is them^{th} derivative of the unknown function,

$f(x)$is the known function of x and $(\delta_0, \delta_1, \delta_2, \ldots, \delta_{m-1}) \in \mathbb{R}$

Taking Complex SEE transform on both side of (4)

$$S^c\{u^{(m)}(x)\} = S^c\{f(x)\} + \lambda S^c \left\{ \int_0^{(x)} K(x,t)\,u(t)dt \right\} \qquad \ldots(5)$$

Applying convolution property of Complex SEE transform on (5)

$$S^c\{u^{(m)}(x)\} = S^c\{f(x)\} + \lambda v^n S^c\{K(x)\}.S^c\{u(x)\} \qquad \ldots(6)$$

Applying Complex SEE Transform property of derivatives of functions on (6)

$$\left[\frac{1}{v^n}[-u^{(m-1)}(0) - (iv)u^{(m-2)}(0) - (iv)^2u^{(m-3)}(0) - \cdots - (iv)^{m-1}u(0)] + (iv)^m S^c\{u(x)\}\right] = S^c\{f(x)\} + \lambda v^n S^c\{K(x)\}.S^c\{u(x)\} \qquad \ldots(7)$$

Putting (4) in (7) we get,

$$\left[\frac{1}{v^n}([-(iv)\delta_{m-2}) - (iv)^2\delta_{m-3} - \cdots - (iv)^{m-1}\delta_0] + (iv)^m S^c\{u(x)\}\right] = S^c\{f(x)\} + \lambda v^n S^c\{K(x)\}.S^c\{u(x)\}$$

$$\Rightarrow [(iv)^m + \lambda v^n S^c\{K(x)\}]S^c\{u(x)\} = S^c\{f(x)\} + \left[\frac{1}{v^n}([(iv)\delta_{m-2}) + (iv)^2\delta_{m-3} + \cdots + (iv)^{m-1}\delta_0]\right]$$

$$\Rightarrow\ S^c\{u(x)\} = \frac{=S^c\{f(x)\} + \left[\frac{1}{v^n}([(iv)\delta_{m-2}) + (iv)^2\delta_{m-3} + \cdots + (iv)^{m-1}\delta_0]\right]}{[(iv)^m + \lambda v^n S^c\{K(x)\}]} \qquad \ldots(8)$$

Provided $[(iv)^m + \lambda v^n S^c\{K(x)\}] \neq 0$

Taking inverse Complex SEE transform of (8) from table 2.4 gives the required exact solution of (3) the Volterra integro-differential equation of the second kind.

2.2 Mohand Transform

We considered the Volterra integro-differential equation of the second kind given in (3) with conditions (4)

Taking the Mohand transform of both side of (3), we have

$$M\{u^{(m)}(x)\} = M\{f(x)\} + \lambda M\left\{\int_0^{(x)} K(x,t)\,u(t)dt\right\} \qquad \ldots(9)$$

Applying convolution property of Mohand transform on (9)

$$M\{u^{(m)}(x)\} = M\{f(x)\} + \lambda\frac{1}{v^2}M\{K(x)\}M\{u(x)\} \qquad \ldots(10)$$

Applying Mohand transform property of derivatives of functions on (10)

$$\left[v^n M\{u(x)\} - v^{n+1}u(0) - v^n u'(0) - v^{n-1}u''(0) - \cdots - v^2 u^{(m-1)}(0)\right] = M\{f(x)\} + \lambda\frac{1}{v^2}M\{K(x)\}M\{u(x)\} \qquad \ldots(11)$$

Putting (4) into (11) we get,

$$[v^n M\{u(x)\} - v^{n+1}\delta_0 - v^n\delta_1 - v^{n-1}\delta_2 - \cdots - v^2\delta_{m-1}] = M\{f(x)\} + \lambda\frac{1}{v^2}M\{K(x)\}M\{u(x)\}$$

$$\Rightarrow \left[v^n - \lambda\frac{1}{v^2}M\{K(x)\}\right]M\{u(x)\} = M\{f(x)\} + [v^{n+1}\delta_0 + v^n\delta_1 + v^{n-1}\delta_2 + \cdots + v^2\delta_{m-1}]$$

$$\Rightarrow M\{u(x)\} = \frac{M\{f(x)\} + [v^{n+1}\delta_0 + v^n\delta_1 + v^{n-1}\delta_2 + \cdots + v^2\delta_{m-1}]}{\left[v^n - \lambda\frac{1}{v^2}M\{K(x)\}\right]} \qquad \ldots(12)$$

Provided $\left[v^n - \lambda\frac{1}{v^2}M\{K(x)\}\right] \neq 0$

Taking inverse Mohand transform of (12) from table 2.4 gives the required exact solution of (3) theVolterra integro-differential equation of the second kind.

4. SYSTEMS OF VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

The systems of Volterra integro-differential equations appear in two kinds. For systems of Volterra integral equations of the first kind, the unknown functions appear only under the integral sign while for systems of Volterra integro-differential equations of the second kind, the unknown functions appear inside and outsidethe integral sign. In this research work we will be working with the equation of the first kind in the form:

$$f_1(x) = \int_0^x \left(K_1(x,t)u(t) + \widetilde{K_1}(x,t)v(t) + \cdots\right)dt$$

$$f_2(x) = \int_0^x \left(K_2(x,t)u(t) + \widetilde{K_2}(x,t)v(t) + \cdots\right)dt \qquad \ldots(13)$$

5. APLLICATIONS

In this section we will apply the Complex SEE and Mohand Transforms to solve some Volterra integro-differential equations of the second kind and systems of Volterra integro-differential equation of the first kind.

Example 1: Consider the Volterra integro-differential equation of the second kind:

$$u''(x) = x + \int_0^{(x)} (x-t)\,u(t)dt \qquad \ldots(14)$$

$$u(0) = 0, u'(0) = 1 \qquad \ldots.(15)$$

Using the Complex SEE transform to solve:

Taking Complex SEE transform on both side of (14)

$$S^c\{u''(x)\} = S^c\{x\} + S^c\left\{\int_0^{(x)} (x-t)\,u(t)dt\right\} \qquad \ldots(16)$$

Applying convolution property of Complex SEE transform on (16)

$$S^c\{u''(x)\} = S^c\{x\} + v^n S^c\{x\} S^c\{u(x)\} \quad \text{....}(17)$$

Applying Complex SEE Transform property of derivatives of functions on (17)

$$\left[\frac{-u'(0)}{v^n} - \frac{iu(0)}{v^{n-1}} - v^2 S^c\{u(x)\}\right] = S^c\{x\} + v^n S^c\{x\} S^c\{u(x)\} \quad \text{....}(18)$$

Putting (15) into (18), we get

$$\frac{-1}{v^n} - v^2 S^c\{u(x)\} = \frac{-1}{v^{n+2}} - \frac{v^n}{v^{n+2}} S^c\{u(x)\}$$

$$\Rightarrow \left[\frac{-v^2}{1} + \frac{1}{v^2}\right] S^c\{u(x)\} = \frac{1}{v^n} - \frac{1}{v^{n+2}}$$

$$\Rightarrow \left[\frac{-v^4 + 1}{v^2}\right] S^c\{u(x)\} = \frac{v^2 - 1}{v^{n+2}}$$

$$\Rightarrow \frac{(1 - v^2)(1 + v^2)}{v^2} S^c\{u(x)\} = \frac{v^2 - 1}{v^{n+2}}$$

$$S^c\{u(x)\} = \frac{-1}{v^n(1 + v^2)} \quad \text{...}(19)$$

Taking inverse Complex SEE transform of (19)

$$u(x) = S^{c-1}\left\{\frac{-1}{v^n(1 + v^2)}\right\}$$

$$u(x) = \sinh(x)$$

This gives the exact solution of (14).

Example2:Consider the Volterra integro-differential equation of the second kind given in equation (14) with the initial condition (15).

$$u''(x) = x + \int_0^{(x)} (x - t)\, u(t) dt$$

$$u(0) = 0, u'(0) = 1$$

Using the Mohand transform to solve:

Taking the Mohand transform of both sides, gives

$$M\{u''(x)\} = M\{x\} + M\left\{\int_0^{(x)} (x - t)\, u(t) dt\right\} \quad \text{...}(20)$$

Applying convolution property of Mohand transform on (20)

$$M\{u''(x)\} = M\{x\} + \frac{1}{v^2} M\{x\} M\{u(x)\} \quad \text{...}(21)$$

Applying Mohand transform property of derivatives of functions on (21)

$$[v^2 M\{u(x)\} - v^3 u(0) - v^2 u'(0)] = 1 + \frac{1}{v^2}.1.M\{u(x)\} \quad \text{...}(22)$$

Putting (15) into (22) we get,

$$v^2 M\{u(x)\} - v^2 = 1 + \frac{1}{v^2} M\{u(x)\}$$

$$\Rightarrow \left[\frac{v^2}{1} - \frac{1}{v^2}\right] M\{u(x)\} = v^2 + 1$$

$$M\{u(x)\} = \frac{v^2}{v^2 - 1} \quad \text{...}(23)$$

Taking inverse Mohand transform of (23)

$$u(x) = M^{-1}\left\{\frac{v^2}{v^2 - 1}\right\}$$

$$u(x) = \sinh(x)$$

This gives the required exact solution of equation (14).

We can see that using both Complex SEE and Mohand integral transform on the same Volterra integro-differential equation of the second kind gave the same exact solution.

Example3:Consider the system of Volterra integro-differential equation of the first kind:

$$\frac{1}{2}x^2+\frac{1}{2}x^3+\frac{1}{12}x^4=\int_0^x[(x-t-1)u(t)+(x-t+1)v(t)]dt$$

$$\frac{3}{2}x^2-\frac{1}{6}x^3+\frac{1}{12}x^4=\int_0^x[(x-t+1)u(t)+(x-t-1)v(t)]dt \qquad \ldots(24)$$

Using the Complex SEE transform to solve:

Taking Complex SEE transform of both sides of (24) gives,

$$\frac{1}{2}S^c\{x^2\}+\frac{1}{2}S^c\{x^3\}+\frac{1}{12}S^c\{x^4\}=S^c\left\{\int_0^x(x-t-1)u(t)dt\right\}+S^c\left\{\int_0^x(x-t+1)v(t)dt\right\}$$

$$\frac{3}{2}S^c\{x^2\}-\frac{1}{6}S^c\{x^3\}+\frac{1}{12}S^c\{x^4\}=S^c\{\int_0^x(x-t+1)u(t)dt\}+S^c\{\int_0^x(x-t-1)v(t)dt\}\ldots \qquad (25)$$

Applying convolution property of Complex SEE transform on (25) gives,

$$\frac{1}{2}S^c\{x^2\}+\frac{1}{2}S^c\{x^3\}+\frac{1}{12}S^c\{x^4\}=v^nS^c\{-1\}S^c\{x\}S^c\{u(x)\}+v^nS^c\{1\}S^c\{x\}S^c\{v(x)\}$$

$$\frac{3}{2}S^c\{x^2\}-\frac{1}{6}S^c\{x^3\}+\frac{1}{12}S^c\{x^4\}=v^nS^c\{1\}S^c\{x\}S^c\{u(x)\}+v^nS^c\{-1\}S^c\{x\}S^c\{v(x)\} \qquad \ldots(26)$$

$$\frac{1}{2}\left[\frac{2!\,i}{v^{n+3}}\right]+\frac{1}{2}\left[\frac{3!\,i}{v^{n+4}}\right]+\frac{1}{12}\left[\frac{-4!\,i}{v^{n+5}}\right]=v^n\left[\frac{i}{v^{n+1}}\right]\left[\frac{-1}{v^{n+2}}\right]S^c\{u(x)\}+v^n\left[\frac{-i}{v^{n+1}}\right]\left[\frac{-1}{v^{n+2}}\right]S^c\{v(x)\}$$

$$\frac{3}{2}\left[\frac{2!\,i}{v^{n+3}}\right]-\frac{1}{6}\left[\frac{3!\,i}{v^{n+4}}\right]+\frac{1}{12}\left[\frac{-4!\,i}{v^{n+5}}\right]=v^n\left[\frac{-i}{v^{n+1}}\right]\left[\frac{-1}{v^{n+2}}\right]S^c\{u(x)\}+v^n\left[\frac{i}{v^{n+1}}\right]\left[\frac{-1}{v^{n+2}}\right]S^c\{v(x)\}$$

$$\frac{iv^2+3v-2i}{v^{n+5}}=\frac{-i}{v^3}S^c\{u(x)\}+\frac{i}{v^3}S^c\{v(x)\}$$

$$\frac{3v^2-v-2i}{v^{n+5}}=\frac{i}{v^3}S^c\{u(x)\}+\frac{-i}{v^3}S^c\{v(x)\} \qquad \ldots.(27)$$

Solving (27) simultaneously for $S^c\{u(x)\}$ and $S^c\{v(x)\}$ gives,

$$S^c\{u(x)\}=\left[\frac{-i}{v^{n+1}}-\frac{1}{v^{n+2}}\right], \qquad S^c\{v(x)\}=\left[\frac{-i}{v^{n+1}}-\frac{2i}{v^{n+3}}\right]$$

Taking the Complex SEE inverse we get,

$$u(x)=S^{c-1}\left\{\frac{-i}{v^{n+1}}-\frac{1}{v^{n+2}}\right\}, \qquad v(x)=S^{c-1}\left\{\frac{-i}{v^{n+1}}-\frac{2i}{v^{n+3}}\right\},$$

$$u(x)=1+x, \qquad v(x)=1+x^2$$

This gives the exact solutions of equations in (24).

Example4:Consider the equationsused in *Example 3.*

$$\frac{1}{2}x^2+\frac{1}{2}x^3+\frac{1}{12}x^4=\int_0^x[(x-t-1)u(t)+(x-t+1)v(t)]dt$$

$$\frac{3}{2}x^2-\frac{1}{6}x^3+\frac{1}{12}x^4=\int_0^x[(x-t+1)u(t)+(x-t-1)v(t)]dt$$

Using the Mohand transform to solve:

Taking Mohand transform of both sides gives,

$$\frac{1}{2}M\{x^2\}+\frac{1}{2}M\{x^3\}+\frac{1}{12}M\{x^4\}=M\left\{\int_0^x(x-t-1)u(t)\right\}+M\left\{\int_0^x(x-t+1)v(t)dt\right\}$$

$$\frac{3}{2}M\{x^2\}-\frac{1}{6}M\{x^3\}+\frac{1}{12}M\{x^4\}=M\{\int_0^x(x-t+1)u(t)\}+M\{\int_0^x(x-t-1)v(t)dt\}\ \ldots(28)$$

Applying convolution property of Mohand transform on (28) gives,

$$\frac{1}{2}M\{x^2\}+\frac{1}{2}M\{x^3\}+\frac{1}{12}M\{x^4\}=\frac{1}{v^2}M\{-1\}M\{x\}M\{u(x)\}+\frac{1}{v^2}M\{1\}M\{x\}M\{v(x)\}$$

$$\frac{3}{2}M\{x^2\}-\frac{1}{6}M\{x^3\}+\frac{1}{12}M\{x^4\}=\frac{1}{v^2}M\{1\}M\{x\}M\{u(x)\}+\frac{1}{v^2}M\{-1\}M\{x\}M\{v(x)\}$$

$$\Rightarrow \frac{1}{2}\left[\frac{2!}{v}\right]+\frac{1}{2}\left[\frac{3!}{v^2}\right]+\frac{1}{12}\left[\frac{4!}{v^3}\right]=\frac{1}{v^2}[-v][1]M\{u(x)\}+\frac{1}{v^2}[v][1]M\{v(x)\}$$

$$\frac{1}{2}\left[\frac{2!}{v}\right]+\frac{1}{2}\left[\frac{3!}{v^2}\right]+\frac{1}{12}\left[\frac{4!}{v^3}\right]=\frac{1}{v^2}[-v][1]M\{u(x)\}+\frac{1}{v^2}[v][1]M\{v(x)\}$$

$$\Rightarrow \frac{(v+1)(v+2)}{v^3}=-\frac{1}{v}M\{u(x)\}+\frac{1}{v}M\{v(x)\}$$

$$\frac{(v-1)(3v-2)}{v^3}=\frac{1}{v}M\{u(x)\}-\frac{1}{v}M\{v(x)\} \qquad \dots(29)$$

Solving (29) simultaneously for $M\{u(x)\}$ and $M\{v(x)\}$ we get,

$$M\{u(x)\}=v+1, \qquad M\{v(x)\}=\frac{v^2+2}{v}$$

Taking the inverse Mohand Transform gives,

$$u(x)=M^{-1}\{v+1\}, \qquad v(x)=M^{-1}\left\{\frac{v^2+2}{v}\right\}$$

$$u(x)=1+x, \qquad v(x)=1+x^2$$

This gives the required exact solutions of the equations (24).

7. RESULT DISCUSSION

We have seen the application of the Complex SEE and Mohand Transforms in solving the Volterra integro-differential equation of the second kind. The steps was similar to the one used in solving the equations of the first kind. The four steps used are as follows:

i. Taking the Complex SEE or Mohand transform of the given equation.

ii. Applying the property of convolution.

iii. Applying the property of derivatives of functions

iv. Taking the inverse Complex SEE or Mohand transform which in return gives the solution to the given equation.

All the examples solved, the Complex SEE and Mohand transforms gives the exact solutions to the equations solved and therefore no need for plotting graph. The transforms equally demonstrated its strength in solving the systems of Volterra integro-differential equations as shown in *Example 1 & 2*.

The systems of Volterra integro-differential equations of the first kind was solved simultaneously just as demonstrated in solving the second kind of Volterra integro-differential equations.

8. CONCLUSION

In what follows, we conclude that the Complex SEE and Mohand transforms demonstrated it superiority in solving the Volterra integro-differential equations of the second kind and the systems of Volterra integro-differential equations of the first kind which can be verified numerically.

We therefore suggest that the Complex SEE and Mohand transforms should be used in solving different types of Volterra integro-differential equations of the second kind and systems of Volterra integro-differential equations of the first kind since it is a straight forward method, easy to understand and friendly in applications. We equally suggest that the methods can be used to solve Volterra integro-differential equations of the third kind and can be extended to other systems of all kinds of equations.

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