International Journal of Research Publication and Reviews

Journal homepage: www.ijrpr.com ISSN 2582-7421

A Study on the Hyperbola $y^2 = 72x^2 - 23$

S. Aarthy Thangam[1], M. Swathi[2], M.A. Gopalan[3]

[1]Assistant Professor, Department of Mathematics, Shrimati Indira Gandhi college, Affiliated to Bharathidasan University,Trichy- 620 002, Tamil Nadu, India. Email: aarthythangam@gmail.com

[2]PG Scholar, Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Trichy-620 002, Tamil Nadu, India. Email: mvswathi560@gmail.com

[3]Professor, Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Trichy-620 002, Tamil Nadu, India. Email: mayilgopalan@gmail.com

ABSTRACT:

This paper deals with the problem of determining non-zero distinct integer solutions to the non-homogeneous binary quadratic equation $y^2 = 72x^2 - 23$. A few interesting properties among the solutions are given. Employing the linear combination among the solutions of the given equation, integer solutions for other choices of hyperbolas and parabolas are obtained.

Keywords: Binary quadratic, Pell equation, Negative Pell equation, Hyperbola

2010 Mathematics Subject classification: 11D09

Introduction:

The Binary quadratic equation of the form $y^2 = Dx^2 + 1$, where D is non square positive integer has been studied by various mathematicians for its non-trivial integer solutions when D takes different integral values [1-2]. For an extensive review of various problems, one may refer [3-10]. In this communication, yet another interesting hyperbola given by $y^2 = 72x^2 - 23$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are obtained. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas, parabolas.

METHOD OF ANALYSIS:

The Negative Pell equation representing hyperbola under consideration is

$$y^2 = 72x^2 - 23 \qquad (1)$$

whose smallest positive integer solution is

$$x_0 = 1, y_0 = 7$$

To obtain the other solutions of (1), consider the Pell equation

$$y^2 = 72x^2 + 1$$

whose general solution is given by

$$\tilde{x}_n = \frac{1}{12\sqrt{2}} g_n;$$

$$\tilde{y}_n = \frac{1}{2} f_n$$

where,

$$f_n = (17 + 12\sqrt{2})^{n+1} + (17 - 12\sqrt{2})^{n+1}$$

$$g_n = (17 + 12\sqrt{2})^{n+1} - (17 - 12\sqrt{2})^{n+1}, \qquad n = -1,0,1,2,.....$$

Applying Brahmagupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the other integer solutions of (1) are given by

$$2x_{n+1} = f_n + \frac{7}{6\sqrt{2}} g_n$$

$$2y_{n+1} = 7 f_n + \frac{12}{\sqrt{2}} g_n$$

The recurrence relations satisfied by the solutions x and y are given by

$$x_{n+3} - 34x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 34y_{n+2} + y_{n+1} = 0$$

Some numerical examples of x and y satisfying (1) are given in the Table:1 below:

Table :1 Numerical examples

n	x_n	y_n
0	1	7
1	31	263
2	1053	8935
3	35771	303527
4	1215161	10310983

From the above table, we observe some interesting relations among the solutions which are presented below:

1. x_n, y_n values are odd.

2. **Relations between solutions**

- $x_{n+3} - 34x_{n+2} + x_{n+1} = 0$
- $2y_{n+1} - x_{n+2} + 17x_{n+1} = 0$
- $2y_{n+2} - 17x_{n+2} + x_{n+1} = 0$
- $2y_{n+3} - 577x_{n+2} + x_{n+1} = 0$
- $68y_{n+1} - x_{n+3} + 577x_{n+1} = 0$
- $4y_{n+2} - x_{n+3} + x_{n+1} = 0$
- $68y_{n+3} - 577x_{n+3} + x_{n+1} = 0$
- $y_{n+2} - 144x_{n+1} - 17y_{n+1} = 0$
- $y_{n+3} - 4896x_{n+1} - 577y_{n+1} = 0$
- $17y_{n+3} - 144x_{n+1} - 577y_{n+2} = 0$
- $2y_{n+1} - 17x_{n+3} + 577x_{n+2} = 0$
- $2y_{n+2} - x_{n+3} + 17x_{n+2} = 0$
- $2y_{n+3} - 17x_{n+3} + x_{n+2} = 0$
- $17y_{n+2} - 144x_{n+2} - y_{n+1} = 0$

- $y_{n+3} - 288x_{n+2} - y_{n+1} = 0$
- $y_{n+3} - 144x_{n+3} - 17y_{n+2} = 0$
- $577y_{n+2} - 144x_{n+2} - 17y_{n+1} = 0$
- $577y_{n+3} - 4896x_{n+3} - y_{n+1} = 0$
- $17y_{n+3} - 144x_{n+3} - y_{n+2} = 0$
- $144y_{n+3} - 4896x_{n+3} - y_{n+2} = 0$
- $y_{n+3} - 34y_{n+2} + y_{n+1} = 0$

3. Each of the following expressions represents a Nasty Number

- $\frac{1}{23}\left(263x_{2n+2} - 7x_{2n+3} + 46\right)$
- $\frac{1}{782}\left(8935x_{2n+2} - 7x_{2n+4} + 1564\right)$
- $\frac{1}{23}\left(144x_{2n+2} - 14y_{2n+2} + 46\right)$
- $\frac{1}{391}\left(4464x_{2n+2} - 168y_{2n+3} + 782\right)$
- $\frac{1}{13271}\left(151632x_{2n+3} - 14x_{2n+4} + 26542\right)$
- $\frac{1}{23}\left(8935x_{2n+3} - 263x_{2n+4} + 46\right)$
- $\frac{1}{391}\left(144x_{2n+3} - 526y_{2n+2} + 782\right)$
- $\frac{1}{23}\left(4464x_{2n+3} - 526y_{2n+3} + 46\right)$
- $\frac{1}{391}\left(151632x_{2n+4} - 526y_{2n+4} + 782\right)$
- $\frac{1}{13271}\left(144x_{2n+4} - 17870y_{2n+2} + 26542\right)$
- $\frac{1}{391}\left(4464x_{2n+4} - 17870y_{2n+3} + 782\right)$
- $\frac{1}{276}\left(1819584x_{2n+4} - 214440y_{2n+4} + 552\right)$
- $\frac{1}{69}\left(3y_{2n+3} - 93y_{2n+2} + 138\right)$

- $\frac{1}{4991}\left(160x_{2n+4}-5134y_{2n+2}+9982\right)$
- $\frac{1}{23}\left(31y_{2n+4}-603y_{2n+3}+46\right)$

4. Each of the following expresssions represents a Cubical Integer

- $\frac{1}{23}[263x_{3n+3}-7x_{3n+4}+789x_{n+1}-21x_{n+2}]$
- $\frac{1}{782}[8935x_{3n+3}-7y_{3n+5}+26805x_{n+1}-21x_{n+3}]$
- $\frac{1}{23}[144x_{3n+3}-14y_{3n+3}+432x_{n+1}-42y_{n+1}]$
- $\frac{1}{391}[4464x_{3n+3}-14y_{3n+4}+13392x_{n+1}-42y_{n+2}]$
- $\frac{1}{13271}[151632x_{3n+3}-14y_{3n+5}+454896x_{n+1}-42y_{n+3}]$
- $\frac{1}{23}[8935x_{3n+4}-263x_{3n+5}+26805x_{n+2}-789x_{n+3}]$
- $\frac{1}{391}[144x_{3n+4}-526y_{3n+3}+432x_{n+2}-1578y_{n+1}]$
- $\frac{1}{23}[4464x_{3n+4}-526y_{3n+4}+13392x_{n+1}-1578y_{n+2}]$
- $\frac{1}{391}[151632x_{3n+4}-526y_{3n+5}+454896x_{n+1}-1578y_{n+3}]$
- $\frac{1}{13271}[144x_{3n+5}-17870x_{3n+3}+432x_{n+3}-53610y_{n+1}]$
- $\frac{1}{391}[4464x_{3n+5}-17870y_{3n+4}+13392x_{n+3}-53610y_{n+2}]$
- $\frac{1}{23}[151632x_{3n+5}-17870y_{3n+5}+454896x_{n+3}-53610y_{n+3}]$
- $\frac{1}{23}[y_{3n+4}-31y_{2n+2}+3y_{n+2}-93y_{n+1}]$
- $\frac{1}{782}[y_{3n+5}-1053y_{3n+3}+3y_{n+3}-3159y_{n+1}]$
- $\frac{1}{23}[31x_{3n+5}-1053y_{3n4}+92y_{n+3}-3159y_{n+2}]$

5. Each of the following expressions represents a bi-quadratic integer

- $\frac{1}{23}[263x_{4n+4}-7x_{4n+5}+1052x_{2n+2}-28x_{2n+3}+138]$
- $\frac{1}{782}[8935x_{4n+4}-7x_{4n+6}+35740x_{2n+2}-28x_{2n+4}+4692]$
- $\frac{1}{23}[144x_{4n+4}-168y_{4n+4}+576x_{2n+2}-56y_{2n+2}+138]$
- $\frac{1}{391}[4464x_{4n+4}-14y_{4n+5}+17856x_{2n+2}-56x_{2n+2}+2346]$
- $\frac{1}{39813}[151632x_{4n+4}-14y_{4n+6}+606528x_{2n+2}-56y_{2n+4}+79626]$
- $\frac{1}{23}[8935x_{4n+5}-263x_{4n+6}+35740x_{2n+3}-1052x_{2n+4}+138]$
- $\frac{1}{391}[144x_{4n+5}-526y_{4n+4}+576x_{2n+3}-2104y_{2n+2}+2346]$
- $\frac{1}{23}[4464x_{4n+5}-526y_{4n+5}+17856x_{2n+3}-2104y_{2n+3}+138]$
- $\frac{1}{391}[151632x_{4n+5}-526y_{4n+6}+606528x_{2n+3}-2104y_{2n+4}+2346]$
- $\frac{1}{13271}[144x_{4n+6}-17870y_{4n+4}+576x_{2n+4}-71480y_{2n+4}+179626]$
- $\frac{1}{391}[4464x_{4n+6}-17870y_{4n+5}+17856x_{2n+4}-71480y_{2n+3}+2346]$
- $\frac{1}{23}[151632x_{4n+6}-17870y_{4n+6}+606528x_{2n+4}-71480y_{2n+4}+138]$
- $\frac{1}{23}[y_{4n+5}-31y_{4n+4}+4y_{2n+3}-124y_{2n+2}+138]$
- $\frac{1}{782}[y_{4n+6}-1053y_{4n+4}+4y_{2n+4}-4212y_{2n+2}+4692]$
- $\frac{1}{23}[151632x_{4n+6}-17870y_{4n+6}+606528x_{2n+4}-71480y_{2n+4}+138]$

6. Each of the following expressions represents a Quintic Integer

- $\frac{1}{23}[263x_{5n+5}-7x_{5n+6}+1315x_{3n+3}-35x_{3n+4}+2630x_{n+1}-70x_{n+2}]$
- $\frac{1}{782}[8935x_{5n+5}-7x_{5n+7}+44675x_{3n+3}-35x_{3n+5}+8935x_{n+1}+7x_{n+3}]$

- $\frac{1}{23}[144x_{5n+5}-14y_{5n+5}+720x_{3n+3}-70y_{3n+3}+1440x_{n+1}-140y_{n+1}]$
- $\frac{1}{391}[4464x_{5n+5}-14y_{5n+6}+22320x_{3n+3}-70y_{3n+4}+44640x_{n+1}-140y_{n+2}]$
- $\frac{1}{13271}[151632x_{5n+5}-14y_{5n+7}+758160x_{3n+3}-70y_{3n+5}+1516320x_{n+1}-140y_{n+3}]$
- $\frac{1}{23}[8935x_{5n+6}-263y_{5n+7}+44675x_{3n+4}-1315x_{3n+5}+89350x_{n+2}-2630x_{n+3}]$
- $\frac{1}{391}[144x_{5n+6}-526y_{5n+5}+720x_{3n+4}-2630y_{3n+3}+1440x_{n+2}-5260y_{n+1}]$
- $\frac{1}{23}[4464x_{5n+6}-526y_{5n+6}+22320x_{3n+4}-2630y_{3n+4}+44640x_{n+2}-5260y_{n+2}]$
- $\frac{1}{391}[151632x_{5n+6}-526y_{5n+7}+758160x_{3n+4}-2630y_{3n+5}+1516320x_{n=2}-5260y_{n+2}]$
- $\frac{1}{13271}[144x_{5n+7}-17870y_{5n+5}+720x_{3n+5}-89350y_{3n+3}+1440x_{n+3}-178700y_{n+1}]$
- $\frac{1}{391}[4464x_{5n+7}-17870y_{5n+6}+22320x_{3n+5}-89350y_{3n+4}+44640x_{n+3}-178700y_{n+2}]$
- $\frac{1}{23}[151632x_{5n+7}-17870y_{5n+7}+758160x_{3n+5}-89350y_{3n+5}+1516320x_{n+3}-178700y_{n+3}]$
- $\frac{1}{23}[y_{5n+6}-31y_{4n+4}+5y_{3n+4}-155y_{2n+2}+10y_{n+2}-310y_{n+1}]$
- $\frac{1}{782}[y_{5n+7}-1053y_{5n+5}+5y_{3n+5}-5265y_{3n+3}+10y_{n+3}+10530y_{n+1}]$
- $\frac{1}{23}[31y_{5n+7}-1053y_{5n+6}+155y_{3n+5}-5265y_{3n+4}+310y_{n+3}-10530y_{n+2}]$

REMARKABLE OBSERVATIONS:

1.Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in the Table: 2 below:

Table: 2 Hyperbolas

S. no	Hyperbolas	(P,Q)
1.	$2P^2-Q^2=2437632$	($P=63125x_{n+1}-168x_{n+2}$, $Q=288x_{n+2}-8928x_{n+1}$)

2.	$2P^2 - Q^2 = 2817902592$	$(p = 214440x_{n+1} - 168x_{n+3}$, $Q = 288x_{n+3} - 303254x_{n+1})$
3.	$2P^2 - Q^2 = 609408$	$(P = 1728x_{n+1} - 168y_{n+1}$, $Q = 288y_{n+2} - 2016x_{n+1})$
4.	$2P^2 - Q^2 = 176118912$	$(P = 53568x_{n+1} - 168y_{n+2}$, $Q = 288y_{n+2} - 75744x_{n+1})$
5.	$2P^2 - Q^2 = 1408955528$	$(P = 151632x_{n+1} - 14y_{n+3}$, $Q = 24y_{n+3} - 214440x_{n+1})$
6..	$2P^2 - Q^2 = 38088$	$(P = 26805x_{n+2} - 789x_{n+3}$, $Q = 1116x_{n+3} - 37908x_{n+2})$
7..	$2P^2 - Q^2 = 1223048$	$(P = 144x_{n+2} - 526y_{n+1}$, $Q = 744y_{n+1} - 168x_{n+2})$
8.	$2P^2 - Q^2 = 4232$	$(P = 4464x_{n+2} - 526y_{n+2}$, $Q = 744y_{n+2} - 6312x_{n+2})$
9.	$2P^2 - Q^2 = 1223048$	$(P = 151632x_{n+2} - 526y_{n+3}$, $Q = 744y_{n+2} - 214440x_{n+1})$
10.	$2P^2 - Q^2 = 1408955528$	$(P = 144x_{n+3} - 17870y_{n+1}$, $Q = 25272y_{n+1} - 168x_{n+3})$
11.	$2P^2 - Q^2 = 1223048$	$(P = 4464x_{n+3} - 17870y_{n+2}$, $Q = 25272y_{n+2} - 6312x_{n+3})$
12.	$2P^2 - Q^2 = 609408$	$(p = 1819584x_{n+3} - 214440y_{n+3}$, $Q = 303264y_{n+3} - 2573280x_{n+3})$
13.	$2P^2 - Q^2 = 49362048$	$(P = 108y_{n+2} - 126y_{n+1}$,

		$Q = 4734y_{n+1} - 126y_{n+2}$)
14.	$2P^2 - Q^2 = 176118912$	($P = 6y_{n+3} - 6318y_{n+1}$, $Q = 8985y_{n+1} - 7y_{n+3}$)
15.	$2P^2 - Q^2 = 152352$	($P = 186y_{n+3} - 6318y_{n+2}$, $Q = 8935y_{n+2} - 263y_{n+3}$)

2. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in the Table: 3 below:

Table: 3 Parabolas

S. no	**Parabolas**	**(R, Q)**
1.	$104R - Q^2 = 2437632$	($R = 6312x_{2n+2} - 168x_{2n+3} + 1104$, $Q = 288x_{n+2} - 8925x_{n+1}$)
2.	$137536R - Q^2 = 2817902592$	($R = 214440x_{2n+2} - 168x_{2n+2} + 37536$, $Q = 288x_{n+3} - 303264x_{n+1}$)
3.	$552R - Q^2 = 609408$	($R = 1728x_{2n+2} - 168y_{2n+3} + 552$, $Q = 288y_{n+1} - 2016x_{n+1}$)
4.	$9384R - Q^2 = 176115912$	($R = 535648x_{2n+2} - 168y_{2n+3} + 9384$, $Q = 288y_{n+2} - 75744x_{n+1}$)
5.	$26542R - Q^2 = 1405955528$	($R = 151632x_{2n+2} - 14y_{2n+4} + 26542$, $Q = 24y_{n+3} - 214440x_{n+1}$)
6.	$138R - Q^2 = 38088$	($R = 2680x_{2n+3} - 789x_{2n+4} + 138$, $Q = 1116x_{n+3} - 37908x_{n+2}$)
7.	$144R - Q^2 = 1223048$	($R = 144x_{2n+3} - 526y_{2n+2} + 782$, $Q = 744y_{n+1} - 168x_{n+2}$)
8.	$46R - Q^2 = 4232$	($R = 4464x_{2n+3} - 526y_{2n+3} + 46$, $Q = 744y_{n+1} - 6312x_{n+2}$)
9.	$732R - Q^2 = 1223048$	($R = 151632x_{2n+3} - 526y_{2n+4} + 782$, $Q = 744y_{n+2} - 214440x_{n+2}$)

10.	$26542R - Q^2 = 1408955528$	$(R = 144x_{2n+4} - 17870y_{2n+2} + 26542,$ $Q = 25272y_{n+1} - 168x_{n+3})$
11.	$782R - Q^2 = 1223048$	$(R = 4464x_{2n+4} - 17870y_{2n+3} + 782,$ $Q = 25272y_{n+2} - 6312x_{n+3})$
12.	$552R - Q^2 = 609408$	$(R = 1819584x_{2n+4} - 214440y_{2n+4} + 552,$ $Q = 303264y_{n+3} - 2573280x_{n+3})$
13.	$4968R - Q^2 = 49362048$	$(R = 108y_{2n+3} - 3348y_{2n+2} + 4968,$ $Q = 4734y_{n+1} - 126y_{n+2})$
14.	$9384R - Q^2 = 176115912$	$(R = 6y_{2n+4} - 6318y_{2n+2} + 9384,$ $Q = 8985y_{n+1} - 7y_{n+3})$
15.	$276R - Q^2 = 152352$	$(R = 186y_{2n+4} - 6318y_{2n+3} + 276,$ $Q = 8935y_{n+2} - 263y_{n+3})$

Conclusion:

In this paper, we have presented infinitely many integer solutions for the hyperbola represented by the negative pell equation $y^2 = 72x^2 - 23$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of pell equations and determine their integer solutions along with suitable properties.

References:

David M. Burton, Elementary Number Theory, Tata MC Graw hill Publishing company, limited New Delhi -2000.

Telang S.J, Number Theory, Tata Mc Graw Hill publishing, Company limited New Delhi-2000.

M.A. Gopalan, S. Aarthy Thangam, S. Selvarani, “Integer solutions on the Hyperbola $y^2 = 87x^2 + 13$”, Asian Journal of Applied science and Technology (AJAST), Volume 1, Issue 6, Page No: 41-44, July 2017.

S. Vidhyalakshmi, T. Mahalakshmi, “On the Negative Pell Equation $y^2 = 15x^2 - 14$”, IJRASET, Volume 7, Issue III, Page No: 891-897, March 2019.

Dr. J. Shanthi, Dr. M. A. Gopalan, “A study on the positive pell equation $y^2 = 42x^2 + 7$”, International Research Journal of Education and Technology, Volume 01, Issue 05, Page No: 107-118, May 2018.

Dr. J. Shanthi, Ms. V. Bahavathi, Dr. M. A. Gopalan, “On the positive Pell Equation $y^2 = 35x^2 = 29$”, International Journal of Enhanced Research in Science, Technology and Engineering, Volume 10, Issue 4, Page No: 40-48, April 2021.

T. Mahalakshmi, E. Shalini, “On the positive pell equation $x^2 = 42y^2 + 7$”, International Journal of Research Publication and Reviews, Volume 4, No: 1, Page No: 763-767, January 2023..

T. Mahalkshmi, J. Shanthi, S. Madhumitha, “On the negative pell equation $y^2 = 80x^2 - 31$”, International Journal of Research Publication and Reviews, Volume 4, No: 10, Page No: 2178-2187, October 2023.

J. Shanthi, T. Mahalakshmi, S. Hemapriya, “On the negative pell equation $y^2 = 87x^2 - 6$”, International Journal of Research Publication and Reviews, Volume 4 No: 10, Page No: 2337-2345, October 2023.

J. Shanthi, T. Mahalakshmi, K. Ayisha siddika, “On the positive pell equation $y^2 = 87x^2 + 13$”, International Journal of Research Publication and Reviews, Volume 4 No: 10, Page No: 2512-2521, October 2023.