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Observations on the Hyperbola $y^2 = 72x^2 + 72$

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ABSTRACT:

The binary quadratic equation $y^2 = 72x^2 + 72$ is considered for obtaining its integral solutions. A few interesting properties among the solutions are given. Employing the linear combination among the solutions of the given equation, integer solutions for other choices of hyperbolas and parabolas are obtained.

Keywords: Binary quadratic, Pell equation, Positive Pell equation, Hyperbola

2010 Mathematics Subject Classification : 11D09

Introduction:

The Binary quadratic equation of the form $y^2 = Dx^2 + 1$, where D is non-square positive integer has been studied by various mathematicians for non-trivial integer solutions when D takes different integral values [1-2] . For an extensive review of various problems, one may refer [3-10]. In this communication, yet another interesting hyperbola given by $y^2 = 72x^2 + 72$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are obtained. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas, parabolas.

METHOD OF ANALYSIS:

The Positive Pell equation representing hyperbola under consideration is

$$y^2 = 72x^2 + 72 \tag{1}$$

whose smallest positive integer solution is

$$x_0 = 1 \,,\, y_0 = 12$$

To obtain the other solutions of (1), consider the Pell equation

$$y^2 = 72x^2 + 1$$

whose general solution is given by

$$\tilde{x}_n = \frac{1}{12\sqrt{2}} g_n \;;\; \tilde{y}_n = \frac{1}{2} f_n$$

where,

$$f_n = \left(17 + 12\sqrt{2}\right)^{n+1} + \left(17 - 12\sqrt{2}\right)^{n+1},$$

$$g_n = \left(17 + 12\sqrt{2}\right)^{n+1} - \left(17 - 12\sqrt{2}\right)^{n+1} \qquad n = -1, 0, 1, 2, 3$$

Applying Brahmagupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the other integer solutions of (1) are given by

$$2x_{n+1} = f_n + \sqrt{2}g_n$$

$$2y_{n+1} = 12 \ f_n + 6\sqrt{2} \ g_n$$

The recurrence relations satisfied by the solutions x and y are given by

$$12x_{n+3} - 408x_{n+2} + 12x_{n+1} = 0$$

$$12y_{n+3} - 408y_{n+2} + 12y_{n+1} = 0$$

Some numerical examples of x and y satisfying (1) are given in the Table: 1 below:

Table :1 Numerical examples

n	x_n	y_n
0	1	12
1	41	348
2	1393	11820
3	47321	401532
4	1607521	13640268

From the above table, we observe some interesting relations among the solutions which are presented below:

1. x_n, y_n values are always odd and even.

2. **Relations between solutions:**

- $12y_{n+3} - 3462x_{n+2} + 102x_{n+1} = 0$
- $3462y_{n+3} - 6y_{n+1} - 29376x_{n+3} = 0$
- $102y_{n+1} - 102y_{n+3} + 29376x_{n+2} = 0$
- $102y_{n+2} - 6y_{n+1} - 864x_{n+2} = 0$
- $6y_{n+2} - 102y_{n+1} - 864x_{n+1} = 0$
- $408y_{n+3} - 3462x_{n+3} + 6x_{n+1} = 0$
- $29376y_{n+2} - 864y_{n+1} - 864y_{n+3} = 0$
- $102y_{n+3} - 6y_{n+2} - 864x_{n+3} = 0$
- $3462y_{n+2} - 6y_{n+3} + 29376x_{n+1} = 0$
- $12x_{n+3} - 408x_{n+2} + 12x_{n+1} = 0$
- $12y_{n+1} - 6x_{n+2} + 102x_{n+1} = 0$
- $12y_{n+2} - 102x_{n+2} + 6x_{n+1} = 0$
- $408y_{n+1} - 6x_{n+3} + 3462x_{n+1} = 0$

- $12y_{n+1} - 102x_{n+3} + 3462x_{n+2} = 0$
- $408y_{n+2} - 102x_{n+3} + 102x_{n+1} = 0$
- $3462y_{n+2} - 102y_{n+3} + 864x_{n+1} = 0$
- $6y_{n+3} - 102y_{n+2} - 864x_{n+2} = 0$
- $12y_{n+2} - 6x_{n+3} + 102x_{n+2} = 0$
- $12y_{n+3} - 102x_{n+3} + 6x_{n+2} = 0$
- $102y_{n+1} - 3462y_{n+2} + 864x_{n+3} = 0$

3. Each of the following expressions represents a Nasty Number:

- $\frac{1}{6}[x_{2n+3} - 29x_{2n+2} + 12]$
- $\frac{1}{204}[x_{2n+4} - 985x_{2n+2} + 408]$
- $\frac{1}{3}[y_{2n+2} - 6x_{2n+2} + 6]$
- $\frac{1}{51}[y_{2n+3} - 246x_{2n+2} + 102]$
- $\frac{1}{1731}[y_{2n+4} - 8358x_{2n+2} + 3462]$
- $\frac{1}{6}[29x_{2n+4} - 985x_{2n+3} + 12]$
- $\frac{1}{51}[29y_{2n+2} - 6x_{2n+3} + 102]$
- $\frac{1}{3}[29y_{2n+3} - 246x_{2n+3} + 6]$
- $\frac{1}{51}[29y_{2n+4} - 8358x_{2n+3} + 102]$
- $\frac{1}{1731}[985y_{2n+2} - 6x_{2n+4} + 3462]$
- $\frac{1}{51}[985y_{2n+3} - 246x_{2n+4} + 102]$
- $\frac{1}{3}[985y_{2n+4} - 8358x_{2n+4} + 6]$

- $\frac{1}{432}[246\,y_{2n+2} - 6\,y_{2n+3} + 864\,]$
- $\frac{1}{14688}[8358\,y_{2n+2} - 6\,y_{2n+4} + 29376\,]$
- $\frac{1}{432}[8358\,y_{2n+3} - 246\,y_{2n+4} + 864\,]$

4. Each of the following expressions represents a cubical integer:

- $\frac{1}{6}[x_{3n+4} - 29x_{3n+3} + 3x_{n+2} - 87x_{n+1}]$
- $\frac{1}{204}[x_{3n+5} - 985x_{3n+3} + 3x_{n+3} - 2955x_{n+1}]$
- $\frac{1}{3}[y_{3n+3} - 6x_{3n+3} + 3y_{n+1} - 18x_{n+1}]$
- $\frac{1}{51}[y_{3n+4} - 246x_{3n+3} + 3y_{n+2} - 738x_{n+1}]$
- $\frac{1}{1731}[y_{3n+5} - 8358\,x_{3n+3} + 3y_{n+3} - 25074\,x_{n+1}]$
- $\frac{1}{6}[29\,x_{3n+5} - 985\,x_{3n+4} + 87\,x_{n+3} - 2955\,x_{n+2}]$
- $\frac{1}{51}[29y_{3n+3} - 6x_{3n+4} + 87y_{n+1} - 18x_{n+2}]$
- $\frac{1}{3}[29\,y_{3n+4} - 246\,x_{3n+4} + 87y_{n+2} - 738\,x_{n+2}\,]$
- $\frac{1}{51}[29\,y_{3n+5} - 8358\,x_{3n+4} + 87\,y_{n+3} - 25074\,\,x_{n+2}]$
- $\frac{1}{1731}[985y_{3n+3} - 6\,x_{3n+5} + 2955\,y_{n+1} - 18\,x_{n+3}]$
- $\frac{1}{51}[985y_{3n+4} - 246\,x_{3n+5} + 2955\,\,y_{n+2} - 738\,x_{n+3}\,]$
- $\frac{1}{3}[985y_{3n+5} - 8358\,\,x_{3n+5} + 2955\,y_{n+3} - 25074\,\,x_{n+3}\,]$
- $\frac{1}{432}[246y_{3n+3} - 6y_{3n+4} + 738y_{n+1} - 18y_{n+2}]$
- $\frac{1}{14688}[8358\,y_{3n+3} - 6y_{3n+5} + 25074\,y_{n+1} - 18y_{n+3}]$

- $\frac{1}{432}[8358\, y_{3n+4} - 246\, y_{3n+5} + 25074\, y_{n+2} - 738\, y_{n+3}]$

5. Each of the following expressions represents a Biquadratic integer:

- $\frac{1}{6}[x_{4n+5} - 29\, x_{4n+4} + 4\, x_{2n+3} - 116\, x_{2n+2} + 36]$
- $\frac{1}{204}[x_{4n+6} - 985\, x_{4n+4} + 4\, x_{2n+4} - 3940\, x_{2n+2} + 1224]$
- $\frac{1}{3}[y_{4n+4} - 6x_{4n+4} + 4y_{2n+2} - 24x_{2n+2} + 18]$
- $\frac{1}{51}[y_{4n+5} - 246x_{4n+4} + 4y_{2n+3} - 984x_{2n+2} + 306]$
- $\frac{1}{1731}[y_{4n+6} - 8358x_{4n+4} + 4y_{2n+4} - 33432\, x_{2n+2} + 10386]$
- $\frac{1}{6}[29\, x_{4n+6} - 985\, x_{4n+5} + 116\, x_{2n+4} - 3940\, x_{2n+3} + 36]$
- $\frac{1}{51}[29\, y_{4n+4} - 6\, x_{4n+5} + 116\, y_{2n+2} - 24\, x_{2n+3} + 306]$
- $\frac{1}{3}[29y_{n+2} - 246x_{n+2} + 116y_{2n+3} - 984x_{2n+3} + 18]$
- $\frac{1}{51}[29y_{4n+6} - 8358\, x_{4n+5} + 116y_{2n+4} - 33432\, x_{2n+3} + 306]$
- $\frac{1}{1731}[985y_{4n+4} - 6x_{4n+6} + 3940y_{2n+2} - 24x_{2n+4} + 10386]$
- $\frac{1}{51}[985y_{4n+5} - 246x_{4n+6} + 3940y_{2n+3} - 984x_{2n+4} + 306]$
- $\frac{1}{3}[985y_{4n+6} - 8358\, x_{4n+6} + 3940y_{2n+4} - 33432\, x_{2n+4} + 18]$
- $\frac{1}{432}[246y_{4n+4} - 6y_{4n+5} + 984y_{2n+2} - 24y_{2n+3} + 2592]$
- $\frac{1}{14688}[8358y_{4n+4} - 6y_{4n+6} + 33432y_{2n+2} - 24y_{2n+4} + 88128]$
- $\frac{1}{432}[8358y_{4n+5} - 246y_{4n+6} + 33432y_{2n+3} - 984y_{2n+4} + 2592]$

6. Each of the following expression represents a Quintic integer:

- $\frac{1}{6}[x_{5n+6}-29x_{5n+5}+5x_{3n+4}-145x_{3n+3}+10x_{n+2}-290x_{n+1}]$
- $\frac{1}{204}[x_{5n+7}-985x_{5n+5}+5x_{3n+5}-4925x_{3n+3}+10x_{n+3}-9850x_{n+1}]$
- $\frac{1}{3}[y_{5n+5}-6x_{5n+5}+5y_{3n+3}-30x_{3n+3}+10y_{n+1}-60x_{n+1}]$
- $\frac{1}{51}[y_{5n+6}-246x_{5n+5}+5y_{3n+4}-1230x_{3n+3}+10y_{n+2}-2460x_{n+1}]$
- $\frac{1}{1731}[y_{5n+7}-8358x_{5n+5}+5y_{3n+5}-41790x_{3n+3}+10y_{n+3}-83580x_{n+1}]$
- $\frac{1}{6}[29x_{5n+7}-985x_{5n+6}+145x_{3n+5}-4925x_{3n+4}+290x_{n+3}-9850x_{n+2}]$
- $\frac{1}{51}[29y_{5n+5}-6x_{5n+6}+145y_{3n+3}-30x_{3n+4}+290y_{n+1}-60x_{n+2}]$
- $\frac{1}{3}[29y_{5n+6}-246x_{5n+6}+145x_{3n+4}-1230x_{3n+4}+290y_{n+2}-2460x_{n+2}]$
- $\frac{1}{51}[29y_{5n+7}-8358x_{5n+6}+145y_{3n+5}-41790x_{3n+4}+290y_{n+3}-83580x_{n+2}]$
- $\frac{1}{1731}[985y_{5n+5}-6x_{5n+7}+4925y_{3n+3}-30x_{3n+5}+9850y_{n+1}-60x_{n+3}]$
- $\frac{1}{51}[985y_{5n+6}-246x_{5n+7}+4925y_{3n+4}-1230x_{3n+5}+9850y_{n+2}-2460x_{n+3}]$
- $\frac{1}{3}[985y_{5n+7}-8358x_{5n+7}+4925y_{3n+5}-41790x_{3n+5}+9850y_{n+3}-83580x_{n+3}]$
- $\frac{1}{432}[246y_{5n+5}-6y_{5n+6}+1230y_{3n+3}-30y_{3n+4}+2460y_{n+1}-60y_{n+2}]$
- $\frac{1}{14688}[8358y_{5n+5}-6y_{5n+7}+41790y_{3n+3}-30y_{3n+5}+83580y_{n+1}-60y_{n+3}]$
- $\frac{1}{432}[8358y_{5n+6}-246y_{5n+7}+41790y_{3n+4}-1230y_{3n+5}+83580y_{n+2}-2460y_{n+3}]$

REMARKABLE OBSERVATIONS:

1. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in the Table: 2 below

Table: 2 Hyperbolas

S. No	Hyperbolas	(P,Q)
1	$2p^2 - Q^2 = 1152$	$(P = 2x_{n+2} - 58x_{n+1}, Q = 82x_{n+1} - 2x_{n+2})$
2	$2P^2 - Q^2 = 1331712$	$(P = 2x_{n+3} - 1970x_{n+1}, Q = 2786x_{n+1} - 2x_{n+3})$
3	$2P^2 - Q^2 = 288$	$(P = 2y_{n+1} - 12x_{n+1}, Q = 24x_{n+1} - 2y_{n+1})$
4	$2P^2 - Q^2 = 83232$	$(P = 2y_{n+2} - 492x_{n+1}, Q = 696x_{n+1} - 2y_{n+2})$
5	$2P^2 - Q^2 = 23970888$	$(P = y_{n+3} - 8358x_{n+1}, Q = 11820x_{n+1} - y_{n+3})$
6	$2P^2 - Q^2 = 1152$	$(P = 58x_{n+3} - 1970x_{n+2}, Q = 2786x_{n+2} - 82x_{n+3})$
7	$2P^2 - Q^2 = 83232$	$(P = 58y_{n+1} - 12x_{n+2}, Q = 24x_{n+2} - 82y_{n+1})$
8	$2P^2 - Q^2 = 288$	$(P = 58y_{n+2} - 492x_{n+2}, Q = 696x_{n+2} - 82y_{n+2})$
9	$2P^2 - Q^2 = 83232$	$(P = 58y_{n+3} - 16716x_{n+2}, Q = 23640x_{n+2} - 82y_{n+3})$
10	$2P^2 - Q^2 = 23970888$	$(P = 985y_{n+1} - 6x_{n+3}, Q = 12x_{n+3} - 1393y_{n+1})$
11	$2P^2 - Q^2 = 83232$	$(p = 1970y_{n+2} - 492x_{n+3}, Q = 696x_{n+3} - 2786y_{n+2})$
12	$2P^2 - Q^2 = 288$	$(P = 1970y_{n+3} - 16716x_{n+3}, Q = 23640x_{n+3} - 2786y_{n+3})$
13	$2P^2 - Q^2 = 5971968$	$(P = 492y_{n+1} - 12y_{n+2}, Q = 24y_{n+2} - 696y_{n+1})$

14	$2P^2 - Q^2 = 1725898752$	$(P = 8358y_{n+1} - 6y_{n+3}, Q = 12y_{n+3} - 11800y_{n+1})$
15	$2P^2 - Q^2 = 1492992$	$(P = 8358y_{n+2} - 246y_{n+3}, Q = 348y_{n+3} - 11820y_{n+2})$

2. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in the Table: 3 below:

Table :3 Parabolas

S. No	**Parabolas**	**(R,Q)**
1	$48R - Q^2 = 1152$	$(R = x_{2n+3} - 29x_{2n+2} + 12, Q = 82x_{n+1} - 2x_{n+2})$
2	$1632R - Q^2 = 1331712$	$(R = x_{2n+4} - 985x_{2n+2} + 408, Q = 2786x_{n+1} - 2x_{n+3})$
3	$24R - Q^2 = 288$	$(R = y_{2n+2} - 6x_{2n+2} + 6, Q = 24x_{n+1} - 2y_{n+1})$
4	$408R - Q^2 = 83232$	$(R = y_{2n+3} - 246x_{2n+2} + 102, Q = 696x_{n+1} - 2y_{n+2})$
5	$3462R - Q^2 = 23970888$	$(R = y_{2n+4} - 8358x_{2n+2} + 3462, Q = 11820x_{n+1} - y_{n+3})$
6	$48R - Q^2 = 1152$	$(R = 29x_{2n+4} - 985x_{2n+3} + 12, Q = 2786x_{n+2} - 82x_{n+3})$
7	$408R - Q^2 = 83232$	$(R = 29y_{2n+2} - 6x_{2n+3} + 102, Q = 24x_{n+2} - 82y_{n+1})$
8	$24R - Q^2 = 288$	$(R = 29y_{2n+3} - 246x_{2n+3} + 6, Q = 696x_{n+2} - 82y_{n+2})$
9	$408R - Q^2 = 83232$	$(R = 29y_{2n+4} - 8358x_{2n+3} + 102, Q = 23640x_{n+2} - 82y_{n+3})$
10	$3462R - Q^2 = 23970888$	$(R = 985y_{2n+2} - 6x_{2n+4} + 3462, Q = 12x_{n+3} - 1393y_{n+1})$

11	$408R - Q^2 = 83232$	$(R = 985y_{2n+3} - 246x_{2n+4} + 102$, $Q = 696x_{n+3} - 2786y_{n+2})$
12	$24R - Q^2 = 288$	$(R = 985y_{2n+4} - 8358x_{2n+4} + 6$, $Q = 23640x_{n+3} - 2786y_{n+3})$
13	$3456R - Q^2 = 5971968$	$(R = 246y_{2n+2} - 6y_{2n+3} + 864$, $Q = 24y_{n+2} - 696y_{n+1})$
14	$29376R - Q^2 = 1725898752$	$(R = 8358y_{2n+2} - 6y_{2n+4} + 29376$, $Q = 12y_{n+3} - 11820y_{n+1})$
15	$864R - Q^2 = 1492992$	$(R = 8358y_{2n+3} - 246y_{2n+4} + 864$, $Q = 348y_{n+3} - 11820y_{n+2})$

CONCLUSION:

In this paper, we have presented infinitely many integer solutions for the hyperbola represented by the positive Pell equation $y^2 = 72x^2 + 72$. As the binary quadratic diophantine equations are rich in variety, one may search for the other choices of Pell equations and determine their integer solutions along with suitable properties.

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