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Application of Lagrange's Multiplier: Minimum Surface Area of a Conical Biscuit Containing Ice Cream and Minimum Surface Area of a Tent for Storage

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ABSTRACT

A conical biscuit is filled with ice cream on the top of which is put also some ice cream in shape of a spherical cap. With given quantity of the ice cream, by applying Lagrange's Multiplier minimum surface area of the biscuit is determined together with the optimum values of radius and height of the cone in form of the biscuit. Thereafter is dealt with minimum surface area of a tent with given volume of the space occupied by it for storage of sand/cement etc.

INTRODUCTION

Many books of Differential Calculus[1]deal with optimization problems applying Lagrange's Multiplier. Present author SN Maitra[2] published a different problem and its solution applying Lagrange's Multiplier.We cannot but mention of cone ice cream that we eat and specially buy for the children. A biscuit of hollow cone shape is filled with ice cream on the top of which is also mounted some ice cream in shape of a spherical cap before handing it over to the buyer .In this paper utilizing Lagrange's Multiplier, minimum value of surface area of the conical biscuit with given volume of ice cream in the cone is determined. In the second part surface area of the tent whose two slanting identical walls facing each other are of rectangular shape. Obviously, either of the other two opposite sides of the tent has a shape of isosceles triangle .With given volume of the space occupied by the tent, its minimum surface area is determined by use of Lagrange's Multiplier. A Problem without its solution from the textbook[1] is incorporated. It is solved herein. A rectangular tank opened at the top is to hold a given volume of water. It is to find the minimum surface area of the tank. An optimisation problem from exercise [1] is solved herein. Let 2x, 2y and z be the length, breadth and height of the tank so that its volume and surface area are given by

$V=4xyz$

$S=4xy+4z(x+y)$

With Lagrange's Multiplier λ is formed function F:

$F =4xy+4z(x+y)+ \lambda(V - 4xyz)$

$\frac{\delta F}{\delta x} = 4y + 4z - 4\lambda yz \ =0$

$\frac{\delta F}{\delta y} = 4x + 4z - 4\lambda xz \ =0$

$\frac{\delta F}{\delta z} = 4(x + y) - 4\lambda xy \ =0$

Multiplying the first equation by y and the second by x and subtracting or equating the first two equations we arrive at the equality

$x=y$

using which in the third equation is obtained

$\lambda = \frac{2}{x}$

Using this in the above equations one gets

$z = x$

Using which in the first equation

$2x.2xx = V$

$x=\sqrt[3]{\frac{V}{4}}=y$ and $z = \sqrt[3]{\frac{3V}{4}}$

Hence the optimum base is a square and the optimum height is half the side of the tank. The minimum surface area of the tank in consequence of the above equations is given by

$S_{min} = 4xy+4z(x+y) = 4x^2 + 8xz = 12(\sqrt[3]{\frac{3V}{4}})^2$

VOLUME OF THE ICE CREAM AND MINIMUM SURFACE AREA OF THE CONE BISCUIT

Let H and r be height and radius of the conical biscuit which is filled with ice cream and thereafter is put at the top some ice cream in shape of a spherical cap of depth h.

Then volume of the spherical cap after drawing an appropriate figure is calculated as

$v_1 = \int_0^h (2Rx - x^2)dx = \pi(Rh^2 - \frac{1}{3}h^3)$ (1)

If R is the radius of the concerned sphere ,the radius r of the base of the cap equal to that of the cone is determined in line with (1) by geometry as

$r^2 = R^2 - (R-h)^2$

Or, $r^2 = 2Rh - h^2$

Or, $R=\frac{r^2+h^2}{2h}$ (2)

using which in (1) is obtained the volume of the ice cream filled in the cap:

$v_1 = \pi(\frac{r^2\text{h}}{2} + \frac{1}{6}h^3)$ (3)

Volume of the ice cream contained in the cone biscuit is

$v_2 = \pi\frac{r^2\text{H}}{3}$ (4)

Adding (3) and (4) we get the total volume V of ice cream in the conical biscuit:

$V = \pi(\frac{r^2\text{H}}{3} + \frac{r^2\text{h}}{2} + \frac{1}{6}h^3)$ *(5)*

Surface area of the conical biscuit is given by

$S = \pi r\sqrt{H^2 + r^2}$ (6)

To find the minimum surface area of the biscuit we choose function F and Lagrange's Multiplier λ such that

$F = \pi r\sqrt{H^2+r^2} + \lambda\,\pi\left(V - \frac{r^2\text{H}}{3} - \frac{r^2\text{h}}{2} - \frac{1}{6}h^3\right)$ (7)

$\frac{\delta F}{\delta r} = \pi\left\{\sqrt{H^2+r^2} + \frac{r^2}{\sqrt{H^2+r^2}} - \frac{2}{3}\lambda r\left(H + \frac{3\text{h}}{2}\right)\right\} = 0$ *(8)*

$\lambda = \frac{3(2r^2+H^2)}{r(2rH+3rh)\sqrt{H^2+r^2}}$ (9)

$\frac{\delta F}{\delta H} = \frac{\pi rH}{\sqrt{H^2+r^2}} - \lambda\,\pi\frac{r^2}{3} = 0$

$\lambda = \frac{3H}{r\sqrt{H^2+r^2}}$ (10)

Equating (9) and (10), we get

$r(2r^2 + H^2) = H(2rH + 3rh)$

0r, $r^2 = \frac{3Hh+H^2}{2}$ (11)

Now we need to determine r and H in terms of given V and as such substituting (11) into (5) is obtained

$V = \pi(\frac{H^2(3\text{h}+\text{H})}{6} + \frac{Hh(3\text{h}+\text{H})}{4} + \frac{1}{6}h^3)$

$= \frac{\pi}{12}(6H^2\text{h} + 2H^3 + 9Hh^2 + 3H^2\text{h} + 2h^3)$

$= \frac{\pi}{12}(9H^2\text{h} + 2H^3 + 9Hh^2 + 2h^3)$ (12)

which is a cubic equation calls for tedious solution to get the values of H in terms of h and V.

Getting the optimum values of H and r by use of the foregoing equations we have

$S_{min} = \pi r_{opt}\sqrt{H_{opt}^2 + r_{opt}{}^2}$ (13)

From realistic point of view in the light of (13) we can choose

Optimum height of the conical biscuit=$H_{opt} = 6 inches$.

Optimum radius of the conical biscuit=$r_{opt} = 1.5\ inches.$

Height of the ice cream in form of spherical cap scooped on the top of the cone= h=2inches. π =3.14.

Then minimum surface area of the biscuit=27.8sq (approx) inches

However, given volume of the ice cream can be calculated by use of (5) and the above numerical values.

MINIMUM SURFACE AREA OF THE TENT AS DEPICTED ABOVE

Let 2x , 2y and z be the length, breadth and height of the tent respectively. Then volume V of the space occupied by the tent and its surface S are respectively given by

V=2xyz (14)

S=4x$\sqrt{y^2+z^2}$+2yz (15)

Involving (14) and (15) with Lagrange's Multiplierλ is formed the working function:

F=4x$\sqrt{y^2+z^2}$+2yz + $\lambda(V-2xyz)$

$\frac{\delta F}{\delta x}=4\sqrt{y^2+z^2}-2\lambda yz$ =0 (16)

$\frac{\delta F}{\delta y}=\frac{4xy}{\sqrt{y^2+z^2}}+2z-2\lambda xz=0$ (17)

$\frac{\delta F}{\delta z}=\frac{4xz}{\sqrt{y^2+z^2}}+2y-2\lambda xy=0$ (18)

Subtracting (18) from (17) is obtained

$\frac{4x(y-z)}{\sqrt{y^2+z^2}}-2(y-z)+2\lambda x(y-z)=0$

Or, y-z=0, or y=z (19)

In consequence of (19) and eliminating λ between (17) and (18),viz, multiplying (16) by x and multiplying (17) by y and thereafter subtracting the latter from the former is obtained

4x$\sqrt{y^2+z^2}$-$\frac{4xy^2}{\sqrt{y^2+z^2}}$-2zy=0 (20)

Simplifying (20),

$\frac{4xz^2}{\sqrt{y^2+z^2}}$-2zy=0

which after squaring both sides leads to

$4x^2z^2$=$y^2(y^2+z^2)$ (21)

which because of inequality (19) gives

$4x^2$=$2y^2=2z^2$ (22)

Or,y=z=$\sqrt{2}$ x (23)

Using relationship (23) in (14), one gets

V=2xyz=$4x^3$

Or,$2x_{opt}$=2$\sqrt[3]{\frac{V}{4}}$, $2y_{opt}$=2$\sqrt{2}.\sqrt[3]{\frac{V}{4}}$, $z_{opt}=\sqrt{2}.\sqrt[3]{\frac{V}{4}}$ (24)

which give the optimum values of length, breadth and height of the tent having minimum surface area , vide equation (14):

$S_{min}=12x_{opt}^2$ =12 $(\sqrt[3]{\frac{V}{4}})^2$ (25)

MINIMUM SURFACE OF A TENT ON A RECTANGULAR BASE

A tent on a rectangular base of length 2x, breadth 2y and of height z, is surmounted by a regular pyramid of height h. If the volume thus enclosed is V, find the minimum surface area of the canvass in the tent. Then

V=4xyz+$\frac{4xyh}{3}$ (26)

where the surface area S is given by

S=4z(x+y)+2x$\sqrt{y^2+h^2}$+2y$\sqrt{x^2+h^2}$ (27)

Let us choose function F and Lagrange's Multiplier λ $utilising$ $(26)and$ (27) such that

F=4z(x+y)+2x$\sqrt{y^2+h^2}$+2y$\sqrt{x^2+h^2}+\lambda\left(\mathrm{V}-4\mathrm{xyz}-\frac{4xyh}{3}\right)$ *(28)*

$\frac{\delta F}{\delta x}=4z+2\sqrt{y^2+h^2}+\frac{2xy}{\sqrt{x^2+h^2}}-4\lambda y\left(\mathrm{z}+\frac{h}{3}\right)=0$ *(29)*

$\frac{\delta F}{\delta y}=4z+2\sqrt{x^2+h^2}+\frac{2xy}{\sqrt{y^2+h^2}}-4\lambda x\left(\mathrm{z}+\frac{h}{3}\right)=0$ *(30)*

$\frac{\delta F}{\delta z}=4(x+y)-4\lambda xy=0$ *(31)*

$$\frac{\delta F}{\delta h}=\frac{2xh}{\sqrt{x^2+h^2}}+\frac{2yh}{\sqrt{y^2+h^2}}-4\lambda\frac{xy}{3}\ =0$$

Combining (29) and (30) is obtained

x=y (32)

which suggests in the light of (31) and (32)

$\lambda=\frac{2}{x}$ (33)

$\frac{4xh}{\sqrt{x^2+h^2}}-4\lambda\frac{x^2}{3}=0$

Or, $\lambda=\frac{3h}{x\sqrt{x^2+h^2}}$ *(34)*

Equating (33) and (34) and simplifying

$9h^2=4(x^2+h^2)$

0r, x=$\frac{\sqrt{5}}{2}h$ =y (35)

From (33) and (35) we get

$\lambda=\frac{4}{h\sqrt{5}}$ *(36)*

Substituting (35) and (36) in either of (29) and (30) is acquired

z=$\frac{h}{2}$ (37)

The above optimum values of x,y,z are substituted in (26) to express them in terms of given volume :

V=$\frac{25h^3}{6}$

Or, h=$\sqrt[3]{\frac{6V}{25}}$ (37.1)

so that by use of the above equations, we get

z=$\frac{\sqrt[3]{\frac{6V}{25}}}{2}$ (38)

x=y=$\frac{\sqrt{5}}{2}\sqrt[3]{\frac{6V}{25}}$ (39)

(38) and (39) yield the optimum values of the variables leading to the minimum values of the surface area given by

S_{min} =5$\sqrt{5}h^2$

MINIMUM SURFACE AREA OF A GARAGE

A garage is constructed of three vertical walls together with a roof at the top and there is no wall along one side y for movement of a car in and out of the garage.

Volume and surface area of the garage are respectively

V=4xyz (40)

S=2(2x+y)z+4xy (41)

where 2x, 2y and z are respectively dimensions of the base and height of the garage. There is no wall along dimension y. Required function F with Lagrange's Multiplierλ is written as

F=2(2x+y)z+4xy +$\lambda(V-4xyz)$ (42)

$\frac{\delta F}{\delta x}=4(z+y)-4\lambda yz=0$

Or, $\lambda=\frac{1}{y}+\frac{1}{z}$ *(43)*

$\frac{\delta F}{\delta y}=2(z+2x)-4\lambda xz=0$

Or, $\lambda=\frac{1}{z}+\frac{1}{2x}$ *(44)*

$\frac{\delta F}{\delta z} = 2(y+2x) - 4\lambda xy$ =0

Or, $\lambda = \frac{1}{2x} + \frac{1}{y}$ *(45)*

Combining (43),(44) and (45) we have

$\frac{1}{y} + \frac{1}{z} = \frac{1}{2x} + \frac{1}{z} = \frac{1}{2x} + \frac{1}{y}$ (46)

which implies

2x=y=z (47)

Using (47) in (40) is obtained

V=4x.2x.2x =16x^3

x= $\sqrt[3]{\frac{V}{16}}$ *y=2* $\sqrt[3]{\frac{V}{16}}$ *=z* *(47)*

represent optimum values of the dimensions of the garage leading to the minimum value of its surface area, vide equation (41):

$S_{min} = 24\sqrt[3]{\frac{V}{16}}$ *(47.1)*

MINIMUM SURFACE AREA OF A CAKE WITH GIVEN VOLUME OF THE STUFF

A cake is in shape of a cylindrical portion of height h and radius r extended by a hemispherical portion having the same radius. Hence volume V and surface area S of the cake are respectively given by

V=$\pi r^2 h + \frac{2\pi}{3} r^3$ (48)

S=$\pi r^2 + 2\pi rh + 2\pi r^2$

$Or, S = 2\pi rh + 3\pi r^2$ (49)

With Lagrange's Multiplierλ $function\ F\ is\ given\ by\ using$ (48)and (49):

F=$2\pi rh + 3\pi r^2 + \lambda\,(V - \pi r^2 h - \frac{2\pi}{3} r^3)$ (50)

$\frac{\delta F}{\delta r} = 2\pi h + 6\pi r - 2\pi\lambda(h+r)r$ =0

Or, $\lambda = \frac{h+3r}{r\,(h+r)}$ (51)

$\frac{\delta F}{\delta h} = 2\pi r - \lambda\,\pi r^2$ =0

$\lambda = \frac{2}{r}$ (52)

Combining (51) and (52) is obtained

h=r (53)

which is substituted in (48) to get

V=$\pi r^3 + \pi \frac{2}{3} r^3$

Or,V=$\frac{5}{3}\pi r^3$

$r_{opt} = \sqrt[3]{\frac{3V}{5\pi}}$ *(54)*

Because of (53),

$h_{opt} = \sqrt[3]{\frac{3V}{5\pi}}$ (54.1)

Using (54) and (54.1) in (48) is obtained the minimum surface area of the cake:

$S_{min} = 5\pi r^2 = 5\pi(\sqrt[3]{\frac{3V}{5\pi}})^2$ (55)

REFERENCE

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