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Laplace Transform and Fractional Calculus Operators Associated with G & I- Function

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ABSTRACT

In the present paper, we discussed the application of Fractional Differential operators Associated with Laplace transform of fractional integral & convolution of G-Function to solve homogenous and non-homogenous linear fractional differential equations using Riemann-Liouville differential operator. Fractional calculus, Laplace transform is a generalization of the Laplace transform both classical sense of the definition as in their properties simplicity, efficiency and the high accuracy.

Fractional Laplace transforms are powerful and efficient techniques for obtaining analytic solution of homogenous and non-homogenous linear fractional differential equations and are in uniformity with the solutions available in the literature.

Keywords - Fractional Differentials operators, convolution, fractional integral G-Function I-Funcion Laplace transform.

AMS 2020 Subject classification No. 26A33, 44A10, 45E10

1. Definition and Introduction

Fractional Calculus is a field of mathematical study that grows out of the classical calculus and being a generalization of classical calculus, it preserves many of its fundamental properties .The fractional calculus plays crucial role in many fields of pure and applied mathematics. One of the most effective methods to solve differential equations is to use integral transforms. Literally, the origination of the integral transforms can be traced back to the work of P.S .Laplace in1780s and Joseph Fourier in1822.Fractional integrals and derivatives, in association with different integral transforms, are widely used to solve different types of differential and integral equations. Fractional differential equations are extensively used in interpretation and modeling in applied mathematics and physics including fluid flow, rheology, electrical circuits, probability and statistics, control theory of dynamical system, viscoelasticity, chemical physics, optics and signal processing and so on[7]. The gist of this paper is to present Fractional Laplace Transform and Our purpose is to extend the application of these methods to obtain the exact solution of homogenous and non-homogenous linear fractional differential equations. The paper is organized as follows. Some necessary definitions and preliminaries of fractional calculus theory are introduced in the Laplace Transform . the object of this paper is to solve an integral equation of convolution form having H- function of two variable as it’s kernel. Some known results are obtained as special cases

beginning. In section 2, we present properties and Lemma’s related to α-Integral Laplace

A new class of convolution integral equations whose kernels involve an H-Function of several variables, which is defined by a multiple contour integral of the Mellin-Barnes type, is solved. It is also indicated how the main theorem can be specialized to derive a number of (known or new) results on convolution integral equations involving simpler special functions of interest in problems of applied mathematics and mathematical physics.

In the present paper a convolution integral equation of Fredholm type whose kernel involves a product of generalized polynomial set, general multivariable polynomials, Fox's H-function and H -function, has been solved by using the theory of Mellin transforms. Our main result is believed to be general and unified in nature. A number of (known and new) results follow as special cases by specializing the coefficients and parameters involved in the kernel.

The following definition and results will be required in this paper

The Laplace Transform if

$$F(P) = L[f(t);p] = \int_0^\infty e^{-pt} f(t)dt \ , \quad \mathrm{Re}(p) > 0 \qquad \ldots(1.1)$$

Then F(p) is called the Laplace transform of f(t) with parameter p and is represented by F(p) = f (t) Erdelyi [(3) pp.129-131]

$$L[f(t);p] = F(P) \text{ then } L[e^{-at}f(t)] = F(p+a) \quad ...(1.2)$$

And if $f(0)=f'(0)=f''(0)=.....=f^{m-1}(0)=0 \quad , f^{n}(t)$

Is continuous and differential, then

$$L[f^{n}(t);p] = P^{n}F(p) \quad ....(1.3)$$

If $L[f_1(t)] = F_1(p)$ then $L[f_2(t)] = F_2(p)$

Then convolution theorem for Laplace transform is

$$L\{\int_0^1 f_1(t)f_2(t-u)du\} = L\{f_1(t)\}L\{f_2(t)\} = F_1(p).F_2(p) \quad ...(1.4)$$

The H-Function Defined by Saxena and kumbhat [1] is an extension of Fox's H-Function on specializing the parameters , H-Function can be reduced to almost all the known special function as well as unknown

The Fox's H-Function of one variable is defined and represented in this Paper as follows [see Srivastava et al [2] ,pp 11-13]

$$H[x] = H_{P,Q}^{M,N}\left[x \Big/ \begin{matrix}(a_j,\alpha_j)_{1,P}\\(b_j,\beta_j)_{1,Q}\end{matrix}\right] = \frac{1}{2\pi\omega}\int_{\theta=N-1}\theta(\xi)x^{\xi}d\xi \quad ...(1.5)$$

$$=-\frac{1}{4\pi^2}\int_{L_1}\int_{L_2}\phi_1(\xi,\eta)\psi_2(\xi)\psi_3(\eta)x(\xi)y(\eta)d\xi d\eta \quad ...(1.6)$$

For condition of the H-Function of one variable (1.5) and on the contour L we refer to srivastava et al [2]

(V) The H-Function of two variable occurring in this paper is defined and represented as follows [see Srivastava et al [2] ,pp 83-85]

$$H[x,y] = H_{p_1,q_1,p_2,q_2,p_3,q_3}^{0,n_1,m_2,n_2,m_3,n_3}\left[x_y \Big/ \begin{matrix}(a_j,\alpha_j,A_j)_{1,p_1} & (c_j,z_j)_{1,p_2} & (e_j,E_j)_{1,p_3}\\(b_j,\beta_j,B_j)_{1,q_1} & (d_j,\delta_j)_{1,q_2} & (f_j,F_j)_{1,q_3}\end{matrix}\right]$$

$$\theta(\xi) = \frac{\prod_{i=1}^{n}\Gamma b_j-\beta_j\xi\prod_{j=1}^{N}\Gamma 1-a_j-\alpha_j\xi}{\prod_{i=M=1}^{Q}\Gamma 1-b_j+\beta_j\xi\prod_{j=N+1}^{P}\Gamma a_j-\alpha_j\xi} \quad ...(1.7)$$

Where

$$\phi_1(\xi,\eta) = \prod_{j=1}^{n_1}\Gamma(1-a_j+\alpha_j\xi+A_j\eta)$$

$$\times\left[\prod_{j=n_1+1}^{p_1}\Gamma a_j-\alpha_j\xi-A_j\eta\prod_{j=1}^{q_1}\Gamma 1-b_j-\beta_j\xi+B_j\eta\right] \quad ...(1.8)$$

Where the $\psi_2(\xi)$ and $\psi_3(\eta)$ are defined as (1.6) and for conditions of existence etc. of the $H(x,y)$ we refer to srivastava et al [2]

(VI) The fractional derivative of special function of one and more variables is important such as in the evaluation of series,[10,15] the derivation of generating function [12,chap.5] and the solution of differential equations [4,14;chap-3] motivated by these and many other avenues of applications, the fractional differential operators $D^{n}_{k,\alpha,x}$ and $_{\alpha}D^{\mu}_{x}$ are much used in the theory of special function of one and more variables .

We use the fractional derivative operator defined in the following manner [9]

$$D^{n}_{k,\alpha,x}(x^{\mu}) = \prod_{r=0}^{n-1}[\frac{\sqrt{\mu+rk+1}}{\sqrt{\mu+rk-\alpha+1}}]x^{\mu+nk} \qquad \text{... (1.9)}$$

Whereα≠μ+1 andα and k are not necessarily integers

(VII) Let α, β *and* η be complex numbers,and let x ∈ R_{+}= (0, ∞) Follwing Saigo [8} Fractional integral (Re(α)>0 and derivative Re(α) < 0 of first kind of a function f(x) on R_{+} are defined respectively in the forms:

$$I^{\alpha,\beta,\eta}_{0,\ x}(\mathrm{f}) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\, 2\mathrm{F1}\left(\alpha+\beta\ ,\ -\eta\ ;\ \alpha\ ;\ 1-\frac{t}{x}\right) f(t)dt\ ; \qquad \text{...(1.10)} \qquad \mathrm{Re}(\alpha)>0$$

$\frac{d^n}{dt^n} I^{\alpha+n\ \beta-n\ ,\eta-n}_{0,\ x}$(f) , 0< Re(α) +n < 1 (n= 1, 2, 3, ……………),

Where 2F1(a, b; c; z) is Gauss's hypergeometric function

Fractional integral Re(α)>0 and derivative Re(α) < 0 of second kind a function f(x) on R_{+} are given by:

$$J^{\alpha,\beta,\eta}_{x,\infty} = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\, t^{\alpha-\beta}\, {}_2F_1\left(\alpha+\beta\ ,\ -\eta\ ;\ \alpha\ ;\ 1-\frac{t}{x}\right) f(t)dt\ ; \qquad \text{....(1.11)} \qquad \mathrm{Re}(\alpha)>0$$

$$= (-1)^n \frac{d^n}{dt^n} j^{\alpha+n\ \beta-n\ ,\eta}_{x,\ \infty}(\mathrm{f}) , \qquad 0< \mathrm{Re}(\alpha) +n < 1 \quad (n= 1, 2, 3, \ldots),$$

Let α, β,η and λ be complex numbers. Then there hold the following formulae. . the R.H.S. has a definite meaning

$$I^{\alpha,\beta,\eta}_{0,\ x}\ t^{\lambda} = \frac{\Gamma(1+\lambda)\Gamma(1+\lambda-\beta+\eta)}{\Gamma(1+\lambda-\beta)\Gamma(1+\lambda+\alpha+\eta)} x^{1-\beta} \qquad \text{.....(1.12)}$$

provided that Re(α)> max [0, Re(β-η)] — 1

(VIII) The I-Function which was recently introduced by saxena [9] is an extension of fox'S H-function. on Specializing the parameters,I-Function can bereduced to almost all the known special function as wellas unknowns. The I-Function of one variable is further studied by so many researchers notably as vaishya, jain, and verma [14], Sharma and shrivastava [12] , Sharma and Tiwari [13] ,Nair [11] with certain properties series summation, integration etc.

Saxena [10] represent and define the I-Function of one variable as follows

$$I\ [\mathrm{Z}] = I^{m,n}_{p_i;\ q_i\ ;\ r}\left[Z \left| \begin{matrix} (a_j, \alpha_j)_{1,n} \ldots \ldots (a_{ji}, \alpha_{ji})_{n+1,\ p_i} \\ \cdot \\ \cdot \\ \cdot \\ (b_j,\ \ \beta_j)_{1,m} \ldots \ldots (b_{ji}, \beta_{ji})_{m+1,q_i} \end{matrix} \right. \right] \qquad \text{......(1.13)}$$

$$= \frac{1}{2\pi i}\int_L t(s)\ z^s\ ds.$$

Where

$$t(s) = \frac{\prod_{J=1}^{m}\Gamma\!\left[b_j - \beta_j s\quad \prod_{J+1}^{n}\Gamma\!\left[1-a_j\ , + \alpha_j s\right.\right.}{\sum_{J=1}\left[\prod_{J=m+1}^{q_i}\Gamma\!\left[1-bji+\beta jis\quad \prod_{J=n+1}^{p_i}\Gamma\!\left[a_{ji} - \alpha_{ji}s\right]\right.\right.} \qquad \text{......(1.14)}$$

The following definition and results will be required in this paper

(IX) Meijer's G-Function and its celebrated generalization the Fox's H-Function [15] are defined in terms of single Meillin Barnes type contour integrals involving quotients of Gamma Function.it is evidenced in the literature.such single Meillin Barnes type contour integrals are useful in finding the analytic solutions of various problems in nuclear and neutrino astrophysics [17].where as the voigt function K(x,y) L(x, y) of astrophysical spectroscopy and of the theory of neutron reactions are expressible as double mellian barnes contour integrals of this same class ([16], [19]).moreover ,the multivariable H-Function contain much more general function as its special cases like generalized Kampe-de-Feriet and generalized lauricella functions (Srivastava-Daust) [18]

2. Main Result

Result I

$$\Rightarrow D^{\eta}_{K,\alpha,x}\{D^{\mu}_{x}(x^{\lambda})\ L\{e^{-nt}\ t^{\eta}\ I^{m,\eta}_{p;,q;r}\left\{\left[zx^{k}\ \middle|\ \begin{matrix}(a_j\ \alpha_j)......(a_{ji}\ \alpha_{ji})p_i\\ \\ (b_j\ \beta_j)......(b_{ji},\beta_{ji})_{qi}\end{matrix}\right];P\right\}$$

$$= x^{\lambda-u+rk^1} \times \frac{\lceil 1+n}{(p+n)^{1+n}}\ I^{m\ \ n+n^1+1}_{p;+n^1+1\ q;+n^1+1\ ;r}$$

$$\left[zx^{k}\ \middle|\ \begin{matrix}(-\lambda,k)\ (-\lambda+u-rk^1,k)\ (a_j\ \alpha_j)......(a_{ji}\ \alpha_{ji})p_i\\ \\ (-\lambda+u)\ (-\lambda+u\mp\alpha-rk^1,k)\ (b_j\ \beta_j)......(b_{ji},\beta_{ji})_{qi}\end{matrix}\right]$$

Provided (in addition to the appropriate convergence and existence conditions)that

λ, μ >0 Re($1+\propto$) > 0 Re(β) > 0 and Re(p) >0 1≥λ>a

Result II

$$L\left\{t^{\alpha}\ G^{mn}_{pq}\left[at^{\lambda}\ \middle|\ \begin{matrix}(a_j)_{1,p}\\(b_j)_{1,q}\end{matrix}\right];P\right. = P^{-1-\alpha}\ G^{m\ \ n+1}_{p;+1,q_1}\left[ap^{-\lambda}\ \middle|\ (-\alpha,\lambda)\begin{matrix}(a_j)_{1,p}\\(b_j)_{1,q}\end{matrix}\right]$$

Provided (in addition to the appropriate convergence and existence conditions)that

λ, μ >0 Re($1+\propto$) > 0 Re(β) > 0 and Re(p) >0 1≥λ>a

Result III

$$\int_0^1 x^{\alpha-1}\ (1-x)^{\beta-1}\ G^{mn}_{pq}\left[z_1x^{\lambda}\ \middle|\ \begin{matrix}(a_j)_{1,p}\\(b_j)_{1,q}\end{matrix}\right]\times G^{gh}_{uv}\left[z_2(1-x)^{u}\ \middle|\ \begin{matrix}(ek)_{1,u}\\(fk)_{1,v}\end{matrix}\right]dx$$

$$= G^{0,0\ \ m,n+1\ \ g,h+1}_{0,1\ p+1,q,\ \ u+1,v}\left[\begin{matrix}z_1\\ \cdot\\ \cdot\\ z_r\end{matrix}\ \middle|\ \begin{matrix}(1-\alpha,\lambda))\ (a_j)_{1,p}(1-\beta,\mu)(ek)_{1,u}\\ \\ (b_j)_{1,q}\ \ (f_j)_{1,v}\end{matrix}\right]$$

Provided (in addition to the appropriate convergence and existence conditions)that

λ, μ >0 Re($1+\propto$) > 0 Re(β) > 0 and Re(p) >0 1≥λ>a

Proof I First Taking Fractional integral formula for I- function for one variable and then using convolution of laplace transform for I-function.

We then apply the formula [7, p.67 eq.4.4.4]

$$D^{\mu}_{x}\left(x^{\lambda}\right)=\frac{\Gamma 1+\lambda}{\Gamma 1+\lambda-\mu}x^{\lambda-\mu},\quad \left[\mathrm{Re}(\lambda)>-1\right]$$

Now Apply the fractional derivative operator defined in the following manner [8]

$$D^{n}_{k,\alpha,x}(x^{\mu})=\prod_{r=0}^{n-1}\left[\frac{\sqrt{\mu+rk+1}}{\sqrt{\mu+rk-\alpha+1}}\right]\mathrm{x}^{\mu+nk}$$

Then useing by mellin barnes type contour integral for I- function for one variable and then convolution of laplace transform for H-function then we get required result

Proof II First using Meijer's G-Function.then by mellin barnes type contour integral for G- function for one variable and get required result. Taking by mellin barnes type contour integral for G- function for one variables and then we get required result.

Proof III First taking product of Meijer's G-Function for one variable. Then using Beta function (Euler's first integral) formula. and then get required result.

3. Conclusion

From this Paper we get some many solution of Fractional Differential operators Associated with Laplace transform of fractional integral & convolution of G-Function

The exact solutions of fractional differential calculus play a crucial role in mathematical physics. Therefore, the validity of the Laplace transform of G-Function, but it requires an observation of the term forcing, so not every fractional differential calculus with a constant coefficient can be solved by the method of Laplace transform. We apply the Laplace transform to fractional integral & convolution of G & I -Function. Analytical solutions of these models for various fractional orders and the solution of the corresponding classical equation were recovered as a particular case. We observe, in the various graphs studied, that the different values of the fractional-order of the derivative allow very different behaviors of the solution, especially in the time of convergence to the equilibrium state, which makes the model convenient to model, among others, growth phenomena.

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5. Reference

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