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On the Positive Pell Equation $y^2 = 87x^2 + 13$

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ABSTRACT:

This paper concerns with the problem of obtaining non-zero distinct integer solutions of the positive pell equation represented by the binary quadratic equation $y^2 = 87x^2 + 13$. A few interesting relations among the solutions are presented. Further by considering suitable linear combinations among the solutions of the considered hyperbola, the other choices of hyperbolas and parabolas.

Keywords: Binary quadratic equation; Hyperbola; Parabola; Integer solutions; pell equation.

2010 mathematics subject classification: 11D09

Introduction:

The binary quadratic equation of the form $y^2 = Dx^2 + 1$ where D is non-square positive integer has been studied by various mathematicians for its non-trivial integer solutions when D takes different integral values [1-4].For an Extensive review of various problems, one may refer [3-11].In this communication, yet another interesting hyperbola given by is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are presented.

METHOD OF ANALYSIS:

The Positive Pell equation representing hyperbola under consideration is

$$y^2 = 87x^2 + 13 \qquad (1)$$

whose smallest positive integer solution is

$$x_0 = 1; y_0 = 10$$

To obtain the other solution of (1), consider the pell equation

$$y^2 = 87x^2 + 1$$

whose general solution is given by

$$\tilde{x}_n = \frac{1}{2\sqrt{87}} g_n \; ; \; \tilde{y}_n = \frac{1}{2} f_n$$

Where

$$f_n = [(28 + 3\sqrt{87})^{n+1} + (28 - 3\sqrt{87})^{n+1}]$$

$$g_n = [(28 - 3\sqrt{87})^{n+1} - (28 - 3\sqrt{87})^{n+1}]$$

Applying Brahmagupta lemma between the other integer solution of (1) are given by

$$x_{n+1} = \frac{1}{2} f_n + \frac{5\sqrt{87}}{87} g_n$$

$$y_{n+1} = 5 f_n + \frac{\sqrt{87}}{2} g_n$$

The recurrence relations satisfied by x and y are given by

$$x_{n+1} - 56x_{n+2} - x_{n+3} = 0$$

$$y_{n+1} - 56y_{n+2} - y_{n+3} = 0$$

A few numerical examples are given in the following table (1)

Table :1 Numerical examples

n	x_n	y_n
0	1	10
1	58	541
2	3247	30286
3	181774	1695475
4	10176097	94916314

From the above table, we observe some interesting properties among the solutions which are presented below:

x_n & y_n values are odd and even.

1. Relations between solutions

- $x_{n+3} - 56x_{n+2} + x_{n+1} = 0$
- $3y_{n+1} - x_{n+2} + 28x_{n+1} = 0$
- $3y_{n+2} - 28x_{n+2} + x_{n+1} = 0$
- $3y_{n+3} - 1567x_{n+2} + 28x_{n+1} = 0$
- $168y_{n+1} - x_{n+3} + 1567x_{n+1} = 0$
- $x_{n+3} - x_{n+1} - 6y_{n+2} = 0$
- $168y_{n+3} - 1567x_{n+3} + x_{n+1} = 0$
- $y_{n+2} - 261x_{n+1} - 28y_{n+1} = 0$
- $y_{n+3} - 1567y_{n+1} - 14616x_{n+1} = 0$
- $28y_{n+3} - 1567y_{n+2} - 261x_{n+1} = 0$
- $3y_{n+1} - 28x_{n+3} + 1567x_{n+2} = 0$

- $3y_{n+2} - x_{n+3} + 28\ x_{n+2} = 0$
- $3y_{n+3} - 28x_{n+3} + x_{n+2} = 0$
- $y_{n+1} + 261\ x_{n+2} - 28\,y_{n+2} = 0$
- $y_{n+3} - y_{n+1} - 522x_{n+2} = 0$
- $y_{n+3} - 28y_{n+2} - 261x_{n+2} = 0$
- $28\,y_{n+1} + 261\,x_{n+3} - 1567\ y_{n+2} = 0$
- $y_{n+1} + 14616\,x_{n+3} - 1567\,y_{n+3} = 0$
- $28\,y_{n+3} - y_{n+2} - 261\,x_{n+3} = 0$
- $y_{n+1} - 56y_{n+2} + y_{n+3} = 0$

2.Each of the following expressions represents a Nasty Number

- $\frac{1}{39}[20x_{2n+3} - 1082x_{2n+2} + 78]$
- $\frac{1}{546}[5x_{2n+4} - 15143x_{2n+2} + 1092]$
- $\frac{1}{13}[20y_{2n+2} - 174x_{2n+2} + 26]$
- $\frac{1}{91}[5y_{2n+3} - 2523x_{2n+2} + 182]$
- $\frac{1}{20371}[20y_{2n+4} - 564978x_{2n+2} + 40742]$
- $\frac{1}{39}[1082x_{2n+4} - 60572x_{2n+3} + 78]$
- $\frac{1}{182}[541y_{2n+2} - 87x_{2n+3} + 364]$
- $\frac{1}{13}[1082y_{2n+3} - 10092x_{2n+3} + 26]$
- $\frac{1}{182}[541\,y_{2n+4} - 282489\ x_{2n+3} + 364]$
- $\frac{1}{20371}[60572\,y_{2n+2} - 174x_{2n+4} + 40742]$
- $\frac{1}{91}[15143\ y_{2n+3} - 2523\ x_{2n+4} + 182]$

- $\frac{1}{13}[60572 y_{2n+4} - 564978 x_{2n+4} + 26]$
- $\frac{1}{39}[116 y_{2n+2} - 2 y_{2n+3} + 78]$
- $\frac{1}{1092}[3247 y_{2n+2} - y_{2n+4} + 2184]$
- $\frac{1}{39}[6494 y_{2n+3} - 116 y_{2n+4} + 78]$

3. Each of the following expressions represents a Cubical Integer

- $\frac{1}{39}[20 x_{3n+4} - 1082 x_{3n+3} + 60 x_{n+2} - 3246 x_{n+1}]$
- $\frac{1}{546}[5 x_{3n+5} - 15143 x_{3n+3} + 15 x_{n+3} - 45429 x_{n+1}]$
- $\frac{1}{13}[20 y_{3n+3} - 174 x_{3n+3} + 60 y_{n+1} - 522 x_{n+1}]$
- $\frac{1}{91}[5 y_{3n+4} - 2523 x_{3n+3} + 15 y_{n+2} - 7569 x_{n+1}]$
- $\frac{1}{20371}[20 y_{3n+5} - 564978 x_{3n+3} + 60 y_{n+3} - 1694934 x_{n+1}]$
- $\frac{1}{39}[1082 x_{3n+5} - 60572 x_{3n+4} + 3246 x_{n+3} - 181716 x_{n+2}]$
- $\frac{1}{182}[541 y_{3n+3} - 87 x_{3n+4} + 1626 y_{n+1} - 261 x_{n+2}]$
- $\frac{1}{13}[1082 y_{3n+4} - 10092 x_{3n+4} + 3246 y_{n+2} - 30276 x_{n+2}]$
- $\frac{1}{182}[541 y_{3n+5} - 282489 x_{3n+4} + 1623 y_{n+3} - 847467 x_{n+2}]$
- $\frac{1}{20371}[60572 y_{3n+3} - 174 x_{3n+5} + 181716 y_{n+1} - 522 x_{n+3}]$
- $\frac{1}{91}[15143 y_{3n+4} - 2523 x_{3n+5} + 45429 y_{n+2} - 7569 x_{n+3}]$
- $\frac{1}{13}[60572 y_{3n+5} - 564978 x_{3n+5} + 181716 y_{n+3} - 1694934 x_{n+3}]$
- $\frac{1}{39}[116 y_{3n+3} - 2 y_{3n+4} + 348 y_{n+1} - 6 y_{n+2}]$

- $\frac{1}{1092}[3247y_{3n+3} - y_{3n+5} + 9741y_{n+1} - 3y_{n+3}]$
- $\frac{1}{39}[6494y_{3n+4} - 116y_{3n+5} + 19482y_{n+2} - 348y_{n+3}]$

4. Each of the following expression represents a Bi-Quadratic Integer

- $\frac{1}{39}[20x_{4n+5} - 1082x_{4n+4} + 80x_{2n+3} - 4328x_{2n+2} + 234]$
- i $\frac{1}{546}[5x_{4n+6} - 15143x_{4n+4} + 20x_{2n+4} - 60572x_{2n+2} + 3276]$
- $\frac{1}{13}[20y_{4n+4} - 174x_{4n+4} + 80y_{2n+2} - 696x_{2n+2} + 522]$
- $\frac{1}{91}[5y_{4n+5} - 2523x_{4n+4} + 20y_{2n+3} - 10092x_{2n+2} + 546]$
- $\frac{1}{20371}[20y_{4n+6} - 564978x_{4n+4} + 80y_{2n+4} - 2259912x_{2n+2} + 488904]$
- $\frac{1}{39}[1082x_{4n+6} - 60572x_{4n+5} + 4328x_{2n+4} - 242288x_{2n+3} + 239]$
- $\frac{1}{182}[541y_{4n+4} - 87x_{4n+5} + 2164y_{2n+2} - 348x_{2n+3} + 1092]$
- $\frac{1}{13}[1082y_{4n+5} - 10092x_{4n+5} + 4328y_{2n+3} - 40368x_{2n+3} + 78]$
- $\frac{1}{182}[541y_{4n+6} - 282489x_{4n+5} + 2164y_{2n+4} - 1129956x_{2n+3} + 1092]$
- $\frac{1}{20371}[60572y_{4n+4} - 174x_{4n+6} + 242288y_{2n+2} - 696x_{2n+4} + 122226]$
- $\frac{1}{13}[60572y_{4n+6} - 564978x_{4n+6} + 242288y_{2n+4} - 2259912x_{2n+4} + 78]$
- $\frac{1}{39}[116y_{4n+4} - 2y_{4n+5} + 464y_{2n+2} - 8y_{2n+3} + 234]$
- $\frac{1}{1092}[3247y_{4n+4} - y_{4n+6} + 12988y_{2n+2} - 4y_{2n+4} + 26208]$
- $\frac{1}{39}[6494y_{4n+5} - 116y_{4n+6} + 25976y_{2n+3} - 464y_{2n+4} + 954]$

5. Each of the following expressions represents a Quintic Integer

- $\frac{1}{39}[20x_{5n+6}-1082x_{5n+5}+100x_{3n+4}-5410x_{3n+3}+200x_{n+2}-10820x_{n+1}]$
- $\frac{1}{546}[5x_{5n+7}-15143x_{5n+5}+25x_{3n+5}-75715x_{3n+3}+50x_{n+3}-151430x_{n+1}]$
- $\frac{1}{13}[20y_{5n+5}-174x_{5n+5}+100y_{3n+3}-870x_{3n+3}+200y_{n+1}-1740x_{n+1}]$
- $\frac{1}{91}[5y_{5n+6}-2523x_{5n+5}+25y_{3n+4}-12615x_{3n+3}-25230x_{n+1}+50y_{n+2}]$
- $\frac{1}{20371}[20y_{5n+7}-564978x_{5n+5}+100y_{3n+5}-2824890x_{3n+3}+200y_{n+3}-564978x_{n+1}]$
- $\frac{1}{39}[1082x_{5n+7}-60572x_{5n+6}+5410x_{3n+5}-302860x_{3n+4}+10820x_{n+3}-605720x_{n+2}]$
- $\frac{1}{182}[541y_{5n+5}-87x_{5n+6}+2705y_{3n+3}-435x_{3n+4}+5425y_{n+1}-870x_{n+2}]$
- $\frac{1}{13}[1082y_{5n+6}-10092x_{5n+6}+5410y_{3n+4}-50460x_{3n+4}+10820y_{n+2}-100920x_{n+2}]$
- $\frac{1}{182}[541y_{5n+1}-282489x_{5n+6}+2705y_{3n+5}-1412445x_{3n+4}+5410y_{n+3}-2824890x_{n+2}]$
- $\frac{1}{20371}[60572y_{5n+5}-174x_{5n+7}+302860y_{3n+3}-870x_{3n+5}+605720y_{n+1}-1740x_{n+3}]$
- $\frac{1}{91}[15143y_{5n+6}-2523x_{5n+7}+75715y_{3n+4}-12615x_{3n+5}+151430y_{n+2}-25230x_{n+3}]$
- $\frac{1}{13}[60572y_{5n+7}-56497x_{5n+7}+302860y_{3n+5}-2824890x_{3n+5}+605720y_{n+3}-5649780x_{n+3}]$
- $\frac{1}{39}[116y_{5n+5}-2y_{5n+6}+580y_{3n+3}-10y_{3n+4}+1160y_{n+1}-20y_{n+2}]$
- $\frac{1}{1092}[3247y_{5n+5}-y_{5n+7}+16235y_{3n+3}-5y_{3n+5}+32470y_{n+1}-10y_{n+3}]$
- $\frac{1}{39}[6494y_{n+2}-116y_{n+3}+32470y_{3n+4}-580y_{3n+5}+64940y_{n+2}-1160y_{n+3}]$

REMARKABLE OBSERVATIONS:

1.Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbola which are presented in the Table: 2 below

Table: 2 Hyperbola

S. no	Hyperbola	(P.Q)
1.	$P^2 - Q^2 = 6084$	$P = [20x_{n+2} - 1082x_{n+1}]$ $Q = \sqrt{87}[116x_{n+1} - 2x_{n+2}]$
2.	$P^2 - Q^2 = 4769856$	$P = [20x_{n+3} - 30286x_{n+1}]$ $Q = \sqrt{87}[3247x_{n+1} - x_{n+3}]$
3.	$P^2 - Q^2 = 676$	$P = [20y_{n+1} - 174x_{n+1}]$ $Q = \sqrt{87}[20x_{n+1} - 2y_{n+1}]$
4.	$P^2 - Q^2 = 132496$	$P = [10y_{n+2} - 5046x_{n+1}]$ $Q = \sqrt{87}[541x_{n+1} - y_{n+2}]$
5.	$P^2 - Q^2 = 1659910564$	$P = [20y_{n+3} - 564978x_{n+1}]$ $Q = \sqrt{87}[60572x_{n+1} - 2y_{n+3}]$
6.	$P^2 - Q^2 = 6084$	$P = [1082x_{n+3} - 60572x_{n+2}]$ $Q = \sqrt{87}[6494x_{n+2} - 116x_{n+3}]$
7.	$P^2 - Q^2 = 132496$	$P = [541y_{n+1} - 87x_{n+2}]$ $Q = \sqrt{87}[10x_{n+2} - 58y_{n+1}]$
8.	$P^2 - Q^2 = 676$	$P = [1082y_{n+2} - 10092x_{n+2}]$ $Q = \sqrt{87}[1082x_{n+2} - 116y_{n+2}]$
9.	$P^2 - Q^2 = 132496$	$P = [541y_{n+3} - 282489x_{n+2}]$ $Q = \sqrt{87}[30286x_{n+2} - 58y_{n+3}]$
10.	$P^2 - Q^2 = 1659910564$	$P = [60572y_{n+1} - 174x_{n+3}]$ $Q = \sqrt{87}[20x_{n+3} - 6494y_{n+1}]$
11.	$P^2 - Q^2 = 132496$	$P = [30286y_{n+2} - 5046x_{n+3}]$ $Q = \sqrt{87}[541x_{n+3} - 3247y_{n+2}]$
12.	$P^2 - Q^2 = 676$	$P = [60572y_{n+3} - 564978x_{n+3}]$ $Q = \sqrt{87}[60572x_{n+3} - 6494y_{n+3}]$

13.	$P^2 - Q^2 = 529308$	$P = [116y_{n+1} - 2y_{n+2}]$ $Q = \frac{1}{\sqrt{87}}[20y_{n+2} - 1082y_{n+1}]$
14.	$P^2 - Q^2 = 414977472$	$P = [3247y_{n+1} - y_{n+3}]$ $Q = \frac{1}{\sqrt{87}}[10y_{n+3} - 30286y_{n+1}]$
15.	$P^2 - Q^2 = 529308$	$P = [6494y_{n+2} - 116y_{n+3}]$ $Q = \frac{1}{\sqrt{87}}[1082y_{n+3} - 60572y_{n+2}]$

2. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabola which are presented in the Table: 3 below:

Table: 3 Parabola

s. no	parabola	(R,Q)
1.	$39R - Q^2 = 6084$	$R = [20x_{2n+3} - 1082x_{2n+2} + 78]$ $Q = \sqrt{87}[116x_{n+1} - 2x_{n+2}]$
2.	$1092R - Q^2 = 4769856$	$R = [10x_{2n+4} - 30286x_{2n+2} + 2184]$ $Q = \sqrt{87}[3247x_{n+1} - x_{n+3}]$
3.	$13R - Q^2 = 676$	$R = [20y_{2n+2} - 174x_{2n+2} + 26]$ $Q = \sqrt{87}[20x_{n+1} - 2y_{n+1}]$
4.	$182R - Q^2 = 132496$	$R = [10y_{2n+3} - 5046x_{2n+2} + 364]$ $Q = \sqrt{87}[541x_{n+1} - y_{n+2}]$
5.	$20371R - Q^2 = 1659910564$	$R = [20y_{2n+4} - 564978x_{2n+2} + 40742]$ $Q = \sqrt{87}[60572x_{n+1} - 2y_{n+3}]$
6.	$39R - Q^2 = 6084$	$R = [1082x_{2n+4} - 60572x_{2n+3} + 78]$ $Q = \sqrt{87}[6494x_{n+2} - 116x_{n+3}]$
7.	$182R - Q^2 = 132496$	$R = [541y_{2n+2} - 87x_{2n+3} + 364]$ $Q = \sqrt{87}[10x_{n+2} - 58y_{n+1}]$
8.	$13R - Q^2 = 676$	$R = [1082y_{2n+3} - 10092x_{2n+3} + 26]$ $Q = \sqrt{87}[1082x_{n+2} - 116y_{n+2}]$
9.	$182R - Q^2 = 132496$	$R = [541y_{2n+4} - 282489x_{2n+3} + 364]$ $Q = \sqrt{87}[30286x_{n+2} - 58y_{n+3}]$

10.	$20371R - Q^2 = 1659910564$	$R = [60572y_{2n+2} - 174x_{2n+4} + 40742]$ $Q = \sqrt{87}[20x_{n+3} - 6494y_{n+1}]$
11.	$182R - Q^2 = 132496$	$R = [30286y_{2n+3} - 5046x_{2n+4} + 364]$ $Q = \sqrt{87}[541x_{n+3} - 3247y_{n+2}]$
12.	$13R - Q^2 = 676$	$R = [60572y_{2n+4} - 564978x_{2n+4} + 26]$ $Q = \sqrt{87}[60572x_{n+3} - 6494y_{n+3}]$
13.	$3393R - Q^2 = 529308$	$R = [116y_{2n+2} - 2y_{2n+3} + 78]$ $Q = \frac{1}{\sqrt{87}}[20y_{n+2} - 1082y_{n+1}]$
14.	$87R - Q^2 = 414977472$	$R = [3247y_{2n+2} - y_{2n+4} + 2184]$ $Q = \frac{1}{\sqrt{87}}[10y_{n+3} - 30286y_{n+1}]$
15.	$3393R - Q^2 = 529308$	$R = [6494y_{2n+3} - 116y_{2n+4} + 78]$ $Q = \frac{1}{\sqrt{87}}[1082y_{n+3} - 60572y_{n+2}]$

CONCLUSION:

In this paper, we have presented infinitely many integer solutions for the hyperbola represented by the positive pell equation $y^2 = 87x^2 + 13$. As the binary quadratic Diophantine equation are rich in variety, one may research for the other choices of pell equations and determine their integer solutions along with suitable properties.

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