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On the Negative Pell Equation $y^2 = 87x^2 - 23$

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ABSTRACT:

The binary quadratic equation represented by the negative Pellian $y^2 = 87x^2 - 23$ is analyzed for its distinct integer solutions. A few interesting relations among the solutions are given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbola, parabola.

Keywords: Binary quadratic, Hyperbola, Parabola, Pell equation, integral solutions.

2010 mathematics subject classification: 11D09

Introduction:

A binary quadratic equation of the form $y^2 = Dx^2 + 1$, where D is non-square positive integer has been studied by various mathematicians for its non-trivial integral solutions when D takes different integral values [1-2]. For an extensive review of various problems, one may refer [3-15]. In this communication, yet another interesting hyperbola given by $y^2 = 87x^2 - 23$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are obtained. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbola, parabola.

METHOD OF ANALYSIS:

The Negative Pell equation representing hyperbola under consideration is

$$y^2 = 87x^2 - 23 \qquad (1)$$

whose smallest positive integer solution is

$$x_0 = 1, y_0 = 8$$

To obtain the other solution of (1), consider the pell equation

$$y^2 = 87x^2 + 1$$

whose general solution is given by

$$\tilde{x}_n = \frac{1}{2\sqrt{87}} g_n, \tilde{y}_n = \frac{1}{2} f_n$$

where

$$f_n = \left(28 + 3\sqrt{87}\right)^{n+1} + \left(28 - 3\sqrt{87}\right)^{n+1}$$

$$g_n = \left(28 + 3\sqrt{87}\right)^{n+1} - \left(28 - 3\sqrt{87}\right)^{n+1}$$

Applying Brahmagupta lemma between $(x_0, y_0) \& (\tilde{x}_n, \tilde{y}_n)$ the other integer solution of (1) are given by

$$x_{n+1} = \frac{1}{2} f_n + \frac{8}{2\sqrt{87}} g_n$$

$$y_{n+1} = \frac{8}{2} f_n + \frac{\sqrt{87}}{2} g_n$$

The recurrence relations satisfied by x and y are given by

$$x_{n+3} - 56x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 56y_{n+2} + y_{n+1} = 0$$

A few numerical examples are given in the following Table:1

Table :1 Numerical examples

n	x_n	y_n
0	1	8
1	52	485
2	2911	27152
3	162964	1520027
4	9123013	85094360

From the above table, we observe some interesting properties among the solutions which are presented below:

x_n & y_n values are odd and even.

1. Relations between solutions

- $x_{n+3} - 56x_{n+2} + x_{n+1} = 0$
- $3y_{n+1} - x_{n+2} + 28x_{n+1} = 0$
- $3y_{n+2} - 28x_{n+2} + x_{n+1} = 0$
- $3y_{n+3} - 1567x_{n+2} + 28x_{n+1} = 0$
- $168y_{n+1} - x_{n+3} + 1567x_{n+1} = 0$
- $6y_{n+2} - x_{n+3} + x_{n+1} = 0$
- $168y_{n+3} - 1567x_{n+3} + x_{n+1} = 0$
- $y_{n+2} - 261x_{n+1} - 28y_{n+1} = 0$
- $y_{n+3} - 14616x_{n+1} - 1567y_{n+1} = 0$
- $28y_{n+3} - 261x_{n+1} - 1567y_{n+2} = 0$
- $3y_{n+2} - x_{n+3} + 28x_{n+2} = 0$
- $3y_{n+3} - 28x_{n+3} + x_{n+2} = 0$
- $28x_{n+3} - 1567x_{n+2} - 3y_{n+1} = 0$
- $y_{n+3} - 522x_{n+2} - y_{n+1} = 0$
- $28y_{n+2} - 261x_{n+2} - y_{n+1} = 0$

- $28y_{n+2} - y_{n+3} + 261x_{n+2} = 0$
- $1567y_{n+2} - 261x_{n+3} - 28y_{n+1} = 0$
- $1567y_{n+3} - 14616x_{n+3} - y_{n+1} = 0$
- $28y_{n+3} - 261x_{n+3} - y_{n+2} = 0$
- $y_{n+3} - 56y_{n+2} + y_{n+1} = 0$

2. Each of the following expressions represents a Nasty Number

- $\frac{1}{69}\left(970x_{2n+2} - 16x_{2n+3} + 138\right)$
- $\frac{1}{483}\left(6788x_{2n+2} - 2x_{2n+4} + 966\right)$
- $\frac{1}{23}\left(174x_{2n+2} - 16y_{2n+2} + 46\right)$
- $\frac{1}{161}\left(2262x_{2n+2} - 4y_{2n+3} + 322\right)$
- $\frac{1}{36041}\left(506514x_{2n+2} - 16y_{2n+4} + 72082\right)$
- $\frac{1}{69}\left(54304x_{2n+3} - 970x_{2n+4} + 138\right)$
- $\frac{1}{322}\left(87x_{2n+3} - 485y_{2n+2} + 644\right)$
- $\frac{1}{23}\left(9048x_{2n+3} - 970y_{2n+3} + 46\right)$
- $\frac{1}{322}\left(253257x_{2n+3} - 485y_{2n+4} + 644\right)$
- $\frac{1}{36041}\left(174x_{2n+4} - 54304y_{2n+2} + 72082\right)$
- $\frac{1}{161}\left(2262x_{2n+4} - 13576y_{2n+3} + 322\right)$
- $\frac{1}{23}\left(506514x_{2n+4} - 54304y_{2n+4} + 46\right)$
- $\frac{1}{69}\left(2y_{2n+3} - 104y_{2n+2} + 138\right)$
- $\frac{1}{1932}\left(y_{2n+4} - 2911y_{2n+2} + 3864\right)$
- $\frac{1}{69}\left(104y_{2n+4} - 5822y_{2n+3} + 138\right)$

3. Each of the following expressions represents a cubical integer

- $\frac{1}{69}\left(970x_{3n+3}-16x_{3n+4}+2910x_{n+1}-48x_{n+2}\right)$
- $\frac{1}{483}\left(6788x_{3n+3}-2x_{3n+5}+20364x_{n+1}-6x_{n+3}\right)$
- $\frac{1}{23}\left(174x_{3n+3}-16y_{3n+3}+522x_{n+1}-48y_{n+1}\right)$
- $\frac{1}{161}\left(2262x_{3n+3}-4y_{3n+4}+6786x_{n+1}-12y_{n+2}\right)$
- $\frac{1}{36041}\left(506514x_{3n+3}-16y_{3n+5}+1519542x_{n+1}-48y_{n+3}\right)$
- $\frac{1}{69}\left(54304x_{3n+4}-970x_{3n+5}+162912x_{n+2}-2910x_{n+3}\right)$
- $\frac{1}{322}\left(87x_{3n+4}-485y_{3n+3}+261x_{n+2}-1455y_{n+1}\right)$
- $\frac{1}{23}\left(9048x_{3n+4}-970y_{3n+4}+27144x_{n+2}-2910y_{n+2}\right)$
- $\frac{1}{322}\left(253257x_{3n+4}-485y_{3n+5}+759771x_{n+2}-1455y_{n+3}\right)$
- $\frac{1}{36041}\left(174x_{3n+5}-54304y_{3n+3}+522x_{n+3}-162912y_{n+1}\right)$
- $\frac{1}{161}\left(2262x_{3n+5}-13576y_{3n+4}+6786x_{n+3}-40728y_{n+2}\right)$
- $\frac{1}{23}\left(506514x_{3n+5}-54304y_{3n+5}+1519542x_{n+3}-162912y_{n+3}\right)$
- $\frac{1}{69}\left(2y_{3n+4}-104y_{3n+3}+6y_{n+2}-312y_{n+1}\right)$
- $\frac{1}{1932}\left(y_{3n+5}-2911y_{3n+3}+3y_{n+3}-8733y_{n+1}\right)$
- $\frac{1}{69}\left(104y_{3n+5}-5822y_{3n+4}+312y_{n+3}-17466y_{n+2}\right)$

4. Each of the following expression represents a bi-quadratic integer

- $\frac{1}{69}\left(970x_{4n+4}-16x_{4n+5}+3880x_{2n+2}-64x_{2n+3}+414\right)$
- $\frac{1}{483}\left(6788x_{4n+4}-2x_{4n+6}+27152x_{2n+2}-8x_{2n+4}+2898\right)$

- $\frac{1}{23}\left(174x_{4n+4}-16y_{4n+4}+696x_{2n+2}-64y_{2n+2}+138\right)$
- $\frac{1}{161}\left(2262x_{4n+4}-4y_{4n+5}+9048x_{2n+2}-16y_{2n+3}+966\right)$
- $\frac{1}{36041}\left(506514x_{4n+4}-16y_{4n+6}+2026056x_{2n+2}-64y_{2n+4}+216246\right)$
- $\frac{1}{69}\left(54304x_{4n+5}-970x_{4n+6}+217216x_{2n+3}-3880x_{2n+4}+414\right)$
- $\frac{1}{322}\left(87x_{4n+5}-485y_{4n+4}+348x_{2n+3}-1940y_{2n+2}+1932\right)$
- $\frac{1}{23}\left(9048x_{4n+5}-970y_{4n+5}+36192x_{2n+3}-3880y_{2n+3}+138\right)$
- $\frac{1}{322}\left(253257x_{4n+5}-485y_{4n+6}+1013028x_{2n+3}-1940y_{2n+4}+1932\right)$
- $\frac{1}{36041}\left(174x_{4n+6}-54304y_{4n+4}+696x_{2n+4}-217216y_{2n+2}+216246\right)$
- $\frac{1}{161}\left(2262x_{4n+6}-13576y_{4n+5}+9048x_{2n+4}-54304y_{2n+3}+966\right)$
- $\frac{1}{23}\left(506514x_{4n+6}-54304y_{4n+6}+2026056x_{2n+4}-217216y_{2n+4}+138\right)$
- $\frac{1}{69}\left(2y_{4n+5}-104y_{4n+4}+8y_{2n+3}-416y_{2n+2}+414\right)$
- $\frac{1}{1932}\left(y_{4n+6}-2911y_{4n+4}+4y_{2n+4}-11644y_{2n+2}+11592\right)$
- $\frac{1}{69}\left(104y_{4n+6}-5822y_{4n+5}+416y_{2n+4}-23288y_{2n+3}+414\right)$

5. Each of the following expressions represents a Quintic integer

- $\frac{1}{69}\left(970x_{5n+5}-16x_{5n+6}+4850x_{3n+3}-80x_{3n+4}+9700x_{n+1}-160x_{n+2}\right)$
- $\frac{1}{483}\left(6788x_{5n+5}-2x_{5n+7}+33940x_{3n+3}-10x_{3n+5}+6788x_{n+1}-20x_{n+3}\right)$
- $\frac{1}{23}\left(174x_{5n+5}-16y_{5n+5}+870x_{3n+3}-80y_{3n+3}+1740x_{n+1}-160y_{n+1}\right)$
- $\frac{1}{161}\left(2262x_{5n+5}-4y_{5n+6}+11310x_{3n+3}-20y_{3n+4}+22620x_{n+1}-40y_{n+2}\right)$
- $\frac{1}{36041}\left(506514x_{5n+5}-16y_{5n+7}+2532570x_{3n+3}-80y_{3n+5}+5065140x_{n+2}-160y_{n+3}\right)$

- $\frac{1}{69}\left(54304x_{5n+6} - 970x_{5n+7} + 271520x_{3n+4} - 4850x_{3n+5} + 543040x_{n+2} - 9700x_{n+3}\right)$
- $\frac{1}{322}\left(87x_{5n+6} - 485y_{5n+5} + 435x_{3n+4} - 2425y_{3n+3} + 870x_{n+2} - 4850y_{n+1}\right)$
- $\frac{1}{23}\left(9048x_{5n+6} - 970y_{5n+6} + 45240x_{3n+4} - 4850y_{3n+4} + 90480x_{n+2} - 9700y_{n+2}\right)$
- $\frac{1}{322}\left(253257x_{5n+6} - 485y_{5n+7} + 1266285x_{3n+4} - 2425y_{3n+5} + 2532570x_{n+2} - 4850y_{n+3}\right)$
- $\frac{1}{36041}\left(174x_{5n+7} - 54304y_{5n+5} + 870x_{3n+5} - 271520y_{3n+3} + 1740x_{n+3} - 543040y_{n+1}\right)$
- $\frac{1}{161}\left(2262x_{5n+7} - 13576y_{5n+6} + 11310x_{3n+5} - 67880y_{3n+4} + 22620x_{n+3} - 13576y_{n+2}\right)$
- $\frac{1}{23}\left(506514x_{5n+7} - 54304y_{5n+7} + 2532570x_{3n+5} - 271520y_{3n+5} + 5065140x_{n+3} - 543040y_{n+3}\right)$
- $\frac{1}{69}\left(2y_{5n+6} - 104y_{5n+5} + 10y_{3n+4} - 520y_{3n+3} + 20y_{n+2} - 1040y_{n+1}\right)$
- $\frac{1}{1932}\left(y_{5n+7} - 2911y_{5n+5} + 5y_{3n+5} - 14555y_{3n+3} + 10y_{n+3} - 29110y_{n+1}\right)$
- $\frac{1}{69}\left(104y_{5n+7} - 5822y_{5n+6} + 520y_{3n+5} - 29110y_{3n+4} + 1040y_{n+3} - 58220y_{n+2}\right)$

REMARKABLE OBSERVATIONS:

1.Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbola which are presented in the Table: 2 below:

Table: 2 Hyperbolas

S. No	Hyperbola	(P,Q)
1	$P^2 - Q^2 = 19044$	$\left(970x_{n+1} - 16x_{n+2},\ \sqrt{87}\left(2x_{n+2} - 104x_{n+1}\right)\right)$
2	$P^2 - Q^2 = 59721984$	$\left(54304x_{n+1} - 16x_{n+3},\ \sqrt{87}\left(2x_{n+3} - 5822x_{n+1}\right)\right)$
3	$P^2 - Q^2 = 2116$	$\left(174x_{n+1} - 16y_{n+1}\ \sqrt{87}\left(2y_{n+1} - 16x_{n+1}\right)\right)$
4	$P^2 - Q^2 = 1658944$	$\left(9048x_{n+1} - 16y_{n+2},\ \sqrt{87}\left(2y_{n+2} - 970x_{n+1}\right)\right)$

5	$P^2 - Q^2 = 5195814724$	$\left(506514x_{n+1} - 16y_{n+3}, \sqrt{87}\left(2y_{n+3} - 54304x_{n+1}\right)\right)$
6	$P^2 - Q^2 = 19044$	$\left(54304x_{n+2} - 970x_{n+3}, \sqrt{87}\left(104x_{n+3} - 5822x_{n+2}\right)\right)$
7	$P^2 - Q^2 = 1658944$	$\left(174x_{n+2} - 970y_{n+1}, \sqrt{87}\left(104y_{n+1} - 16x_{n+2}\right)\right)$
8	$P^2 - Q^2 = 2116$	$\left(9048x_{n+2} - 970y_{n+2}, \sqrt{87}\left(104y_{n+2} - 970x_{n+2}\right)\right)$
9	$P^2 - Q^2 = 1658944$	$\left(506514x_{n+2} - 970y_{n+3}, \sqrt{87}\left(104y_{n+3} - 54304x_{n+2}\right)\right)$
10	$P^2 - Q^2 = 5195814724$	$\left(174x_{n+3} - 54304y_{n+1}, \sqrt{87}\left(5822y_{n+1} - 16x_{n+3}\right)\right)$
11	$P^2 - Q^2 = 1658944$	$\left(9048x_{n+3} - 54304y_{n+2}, \sqrt{87}\left(5822y_{n+2} - 970x_{n+3}\right)\right)$
12	$P^2 - Q^2 = 2116$	$\left(506514x_{n+3} - 54304y_{n+3}, \sqrt{87}\left(5822y_{n+3} - 54304x_{n+3}\right)\right)$
13	$87P^2 - Q^2 = 1656828$	$\left(2y_{n+2} - 104y_{n+1}, \left(970y_{n+1} - 16y_{n+2}\right)\right)$
14	$87P^2 - 64Q^2 = 5195812608$	$\left(2y_{n+3} - 5822y_{n+1}, 6788y_{n+1} - 2y_{n+3}\right)$
15	$87P^2 - Q^2 = 1656828$	$\left(104y_{n+3} - 5822y_{n+2}, 54304y_{n+2} - 970y_{n+3}\right)$

2. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabola which are presented in the Table: 3 below:

Table :3 Parabolas

S. No	**Parabola**	**(R,Q)**
1	$69R - Q^2 = 19044$	$\left(970x_{2n+2} - 16x_{2n+3} + 138, \sqrt{87}\left(2x_{n+2} - 104x_{n+1}\right)\right)$

2	$30912R-Q^2=59721984$	$\begin{pmatrix}6788x_{2n+2}-2x_{2n+4}+966,\\ \sqrt{87}\left(2x_{n+3}-5822x_{n+1}\right)\end{pmatrix}$
3	$23R-Q^2=2116$	$\begin{pmatrix}174x_{2n+2}-16y_{2n+2}+46,\\ \sqrt{87}\left(2y_{n+1}-16x_{n+1}\right)\end{pmatrix}$
4	$2576R-Q^2=1658944$	$\begin{pmatrix}2262x_{2n+2}-4y_{2n+3}+322,\\ \sqrt{87}\left(2y_{n+2}-970x_{n+1}\right)\end{pmatrix}$
5	$36041R-Q^2=5195814724$	$\begin{pmatrix}506514x_{2n+2}-16y_{2n+4}+72082,\\ \sqrt{87}\left(2y_{n+3}-54304x_{n+1}\right)\end{pmatrix}$
6	$69R-Q^2=19044$	$\begin{pmatrix}54304x_{2n+3}-970x_{2n+4}+138\\ \sqrt{87}\left(104x_{n+3}-5822x_{n+2}\right)\end{pmatrix}$
7	$1288R-Q^2=1658944$	$\begin{pmatrix}87x_{2n+3}-485y_{2n+2}+644,\\ \sqrt{87}\left(104y_{n+1}-16x_{n+2}\right)\end{pmatrix}$
8	$23R-Q^2=2116$	$\begin{pmatrix}9048x_{2n+3}-970y_{2n+3}+46,\\ \sqrt{87}\left(104y_{n+2}-970x_{n+2}\right)\end{pmatrix}$
9	$1288R-Q^2=1658944$	$\begin{pmatrix}253257x_{2n+3}-485y_{2n+4}+644,\\ \sqrt{87}\left(104y_{n+3}-54304x_{n+2}\right)\end{pmatrix}$
10	$36041R-Q^2=5195814724$	$\begin{pmatrix}174x_{2n+4}-54304y_{2n+2}+72082\\ \sqrt{87}\left(5822y_{n+1}-16x_{n+3}\right)\end{pmatrix}$
11	$2576R-Q^2=1658944$	$\begin{pmatrix}2262x_{2n+4}-13576y_{2n+3}+322,\\ \sqrt{87}\left(5822y_{n+2}-970x_{n+3}\right)\end{pmatrix}$
12	$23R-Q^2=2116$	$\begin{pmatrix}506514x_{2n+4}-54304y_{2n+4}+46\\ \sqrt{87}\left(5822y_{n+3}-54304x_{n+3}\right)\end{pmatrix}$
13	$6003R-Q^2=1656828$	$\begin{pmatrix}2y_{2n+3}-104y_{2n+2}+138,\\ \left(970y_{n+1}-16y_{n+2}\right)\end{pmatrix}$
14	$168084R-16Q^2=1298953152$	$\begin{pmatrix}y_{2n+4}-2911y_{2n+2}+3864,\\ \left(6788y_{n+1}-2y_{n+3}\right)\end{pmatrix}$

15	$6003R-Q^2=1656828$	$\begin{pmatrix}104y_{2n+4}-5822y_{2n+3}+138, \\ (54304y_{n+2}-970y_{n+3})\end{pmatrix}$

Conclusion:

As Negative Pell equations are rich in variety, one may search for integer

Solutions to other choices of Negative Pell equations.

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