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On the Negative Pell Equation $y^2 = 87x^2 - 6$

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ABSTRACT:

The hyperbola represented by the binary quadratic equation $y^2 = 87x^2 - 6$ is analyzed for finding its non-zero distinct integer solutions. A few interesting relations among its solutions are presented. Also knowing an integral solution of the given hyperbola, integer solution for other choices of hyperbolas and parabolas are presented.

Keywords: Binary quadratic, Hyperbola, Parabola, Pell equation, Integral solutions. 2010 mathematics subject classification: 11D09

Introduction:

A binary quadratic equation of the form $y^2 = Dx^2 + 1$, where D is non-square positive integer has been studied by various mathematicians for its non-trivial integer solutions when D takes different integral values [1-2]. For an extensive review of various problems, one may refer [3-15]. In this communication, yet another interesting hyperbola given by $y^2 = 87x^2 - 6$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are obtained. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas, parabolas.

METHOD OF ANALYSIS:

The Negative Pell equation representing hyperbola under consideration is

$$y^2 = 87x^2 - 6 \qquad (1)$$

whose smallest positive integer solution is

$$x_0 = 1; y_0 = 9$$

To obtain the other solution of (1), consider the pell equation

$$y^2 = 87x^2 + 1$$

whose general solution is given by

$$\tilde{x}_n = \frac{1}{2\sqrt{87}} g_n ; \tilde{y}_n = \frac{1}{2} f_n$$

where

$$f_n = \left(28 + 3\sqrt{87}\right)^{n+1} + \left(28 - 3\sqrt{87}\right)^{n+1}$$

$$g_n = \left(28 + 3\sqrt{87}\right)^{n+1} - \left(28 - 3\sqrt{87}\right)^{n+1}$$

Applying Brahmagupta lemma between (x_0, y_0)&$(\tilde{x}_n, \tilde{y}_n)$ the other integer solution of (1) are given by

$$x_{n+1} = \frac{1}{2} f_n + \frac{9}{2\sqrt{87}} g_n$$

$$y_{n+1} = \frac{9}{2} f_n + \frac{\sqrt{87}}{2} g_n$$

The recurrence relations satisfied by x and y are given by

$$x_{n+3} - 56x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 56y_{n+2} + y_{n+1} = 0$$

A few numerical examples are given in the following Table: 1 below:

Table: 1 Numerical examples

N	x_n	y_n
0	55	513
1	3079	28719
2	172369	1607751
3	9649585	90005337

From the above table, we observe some interesting properties among the solutions which are presented below:

x_n & y_n values are odd.

1. **Relations between solutions**

- $x_{n+1} - 56x_{n+2} + x_{n+3} = 0$
- $28x_{n+1} - x_{n+2} + 3y_{n+1} = 0$
- $28x_{n+1} - 1567x_{n+2} + 3y_{n+3} = 0$
- $168y_{n+1} - x_{n+3} + 1567x_{n+1} = 0$
- $6y_{n+2} - x_{n+3} + x_{n+1} = 0$
- $168y_{n+3} - 1567x_{n+3} + x_{n+1} = 0$
- $y_{n+2} - 261x_{n+1} - 28y_{n+1} = 0$
- $y_{n+3} - 14616x_{n+1} - 1567y_{n+1} = 0$
- $28x_{n+2} - x_{n+1} - 3y_{n+2} = 0$
- $28y_{n+3} - 261x_{n+1} - 1567y_{n+2} = 0$
- $3y_{n+1} - 28x_{n+3} + 1567x_{n+2} = 0$
- $3y_{n+2} - x_{n+3} + 28x_{n+2} = 0$
- $3y_{n+3} - 28x_{n+3} + x_{n+2} = 0$
- $28y_{n+2} - 261x_{n+2} - y_{n+1} = 0$
- $y_{n+3} - 522x_{n+2} - y_{n+1} = 0$

- $y_{n+3} - 261x_{n+2} - 28y_{n+2} = 0$
- $y_{n+2} - 28y_{n+3} + 261x_{n+3} = 0$
- $261x_{n+3} - 1567y_{n+2} + 28y_{n+1} = 0$
- $14616x_{n+3} - 1567y_{n+3} + y_{n+1} = 0$
- $y_{n+1} - 56y_{n+2} + y_{n+3} = 0$

2. **Each of the following expressions represents a Nasty Number**

- $\frac{1}{2}\left(114x_{2n+2} - 2x_{2n+3} + 4\right)$
- $\frac{1}{112}\left(6382x_{2n+2} - 2x_{2n+4} + 224\right)$
- $\frac{1}{6}\left(174x_{2n+2} - 18y_{2n+2} + 12\right)$
- $\frac{1}{168}\left(9570x_{2n+2} - 18y_{2n+3} + 336\right)$
- $\frac{1}{9402}\left(535746x_{2n+2} - 18y_{2n+4} + 18804\right)$
- $\frac{1}{18}\left(57438x_{2n+3} - 1026x_{2n+4} + 36\right)$
- $\frac{1}{168}\left(174x_{2n+3} - 1026y_{2n+2} + 336\right)$
- $\frac{1}{6}\left(9570x_{2n+3} - 1026y_{2n+3} + 12\right)$
- $\frac{1}{168}\left(535746x_{2n+3} - 1026y_{2n+4} + 336\right)$
- $\frac{1}{9402}\left(174x_{2n+4} - 57438y_{2n+2} + 18804\right)$
- $\frac{1}{168}\left(9570x_{2n+4} - 57438y_{2n+3} + 336\right)$
- $\frac{1}{6}\left(535746x_{2n+4} - 57438y_{2n+4} + 12\right)$
- $\frac{1}{18}\left(2y_{2n+3} - 110y_{2n+2} + 36\right)$
- $\frac{1}{1008}\left(2y_{2n+4} - 6158y_{2n+2} + 2016\right)$
- $\frac{1}{18}\left(110y_{2n+4} - 6158y_{2n+3} + 36\right)$

3. **Each of the following expressions represents a cubical integer**

- $\frac{1}{2}(342x_{n+1}-6x_{n+2}+114x_{3n+3}-2x_{3n+4})$
- $\frac{1}{56}(9573x_{n+1}-3x_{n+3}+3191x_{3n+3}-x_{3n+5})$
- $\frac{1}{3}(261x_{n+1}-27y_{n+1}+87x_{3n+3}-9y_{3n+3})$
- $\frac{1}{28}(4785x_{n+1}-9y_{n+2}+1595x_{3n+3}-3y_{3n+4})$
- $\frac{1}{1567}(267873x_{n+1}-9y_{n+3}+89291x_{3n+3}-3y_{3n+5})$
- $\frac{1}{3}(28719x_{n+2}-513x_{n+3}+9573x_{3n+4}-171x_{3n+5})$
- $\frac{1}{28}(29y_{3n+4}-171y_{3n+3}+87x_{n+2}-513y_{n+4})$
- $\frac{1}{2}(9570x_{n+2}-1026y_{n+2}+3190x_{3n+4}-342y_{3n+4})$
- $\frac{1}{28}(267873x_{n+2}-513y_{n+3}+89291x_{3n+4}-171y_{3n+5})$
- $\frac{1}{1567}(87x_{n+3}-28719y_{n+1}+29x_{3n+5}-9573y_{3n+3})$
- $\frac{1}{28}(4785x_{n+3}-28719y_{n+2}+1595x_{3n+5}-9573y_{3n+4})$
- $\frac{1}{3}(803619x_{n+3}-86157y_{n+3}+267873x_{3n+5}-28719y_{3n+5})$
- $\frac{1}{9}(3y_{n+2}-165y_{n+1}+y_{3n+4}-55y_{3n+3})$
- $\frac{1}{504}(3y_{n+3}-9237y_{n+1}+y_{3n+5}-3079y_{3n+3})$
- $\frac{1}{9}(3079y_{3n+4}-9237y_{n+2}+55y_{3n+5}-3069y_{3n+4})$

4. **Each of the following expression represents a bi-quadratic integer**

- $\frac{1}{2}(114x_{4n+4}-2x_{4n+5}+456x_{2n+2}-8x_{2n+3}+12)$
- $\frac{1}{56}(3191x_{4n+4}-x_{4n+6}+12764x_{2n+2}-4x_{2n+4}+336)$
- $\frac{1}{3}(87x_{4n+4}-9y_{4n+4}+348x_{2n+2}-36y_{2n+2}+18)$

- $\frac{1}{28}\left(1595x_{4n+4} - 3y_{4n+5} + 6380x_{2n+2} - 12y_{2n+3} + 168\right)$
- $\frac{1}{1567}\left(89291x_{4n+4} - 3y_{4n+6} + 357164x_{2n+2} - 12y_{2n+4} + 9402\right)$
- $\frac{1}{3}\left(9573x_{4n+5} - 171x_{4n+6} + 38292x_{2n+3} - 684x_{2n+4} + 18\right)$
- $\frac{1}{28}\left(29x_{4n+5} - 171y_{4n+4} + 116x_{2n+3} - 684y_{2n+2} + 168\right)$
- $\frac{1}{3}\left(4785x_{4n+5} - 513y_{4n+5} + 19140x_{2n+3} - 2052y_{2n+3} + 18\right)$
- $\frac{1}{28}\left(89291x_{4n+5} - 171y_{4n+6} + 357164x_{2n+3} - 684y_{2n+4} + 168\right)$
- $\frac{1}{1567}\left(29x_{4n+6} - 9573y_{4n+4} + 116x_{2n+4} - 38292y_{2n+2} + 9402\right)$
- $\frac{1}{28}\left(1595x_{4n+6} - 9573y_{4n+5} + 6380x_{2n+4} - 38292y_{2n+3} + 168\right)$
- $\frac{1}{3}\left(267873x_{4n+6} - 28719y_{4n+6} + 1071492x_{2n+4} - 114876y_{2n+4} + 18\right)$
- $\frac{1}{9}\left(y_{4n+5} - 55y_{4n+4} + 4y_{2n+3} - 220y_{2n+2} + 54\right)$
- $\frac{1}{504}\left(y_{4n+6} - 3079y_{4n+4} + 4y_{2n+4} - 12316y_{2n+2} + 3024\right)$
- $\frac{1}{9}\left(55y_{4n+6} - 3079y_{4n+5} + 220y_{2n+4} - 12316y_{2n+3} + 54\right)$

5. Each of the following expressions represents a Quintic integer

- $\frac{1}{2}\left(114x_{5n+5} - 2x_{5n+6} + 570x_{3n+3} - 10x_{3n+4} + 600x_{n+1} - 20x_{n+2}\right)$
- $\frac{1}{56}\left(3191x_{5n+5} - x_{5n+7} + 15955x_{3n+3} - 5x_{3n+5} + 31910x_{n+1} - 10x_{n+3}\right)$
- $\frac{1}{3}\left(87x_{5n+5} - 9y_{5n+5} + 435x_{3n+3} - 45y_{3n+3} + 870x_{n+1} - 90y_{n+1}\right)$
- $\frac{1}{28}\left(1595x_{5n+5} - 3y_{5n+6} + 7975x_{3n+3} - 15y_{3n+4} + 15950x_{n+1} - 30y_{n+2}\right)$
- $\frac{1}{1567}\left(89291x_{5n+5} - 3y_{5n+7} + 446455x_{3n+3} - 15y_{3n+5} + 892910x_{n+1} - 30y_{n+3}\right)$
- $\frac{1}{3}\left(9573x_{5n+6} - 171x_{5n+7} + 47865x_{3n+4} - 855x_{3n+5} + 9573x_{n+2} - 171x_{n+3}\right)$

- $\frac{1}{28}\left(29x_{5n+6}-171y_{5n+5}+145x_{3n+4}-855y_{3n+3}+290x_{n+2}-1710y_{n+1}\right)$
- $\frac{1}{3}\left(4785x_{5n+6}-513y_{5n+6}+23925x_{3n+4}-2565y_{3n+4}+47850x_{n+2}-5130y_{n+2}\right)$
- $\frac{1}{28}\left(89291x_{5n+6}-171y_{5n+7}+446455x_{3n+4}-855y_{3n+5}+892910x_{n+2}-1710y_{n+3}\right)$
- $\frac{1}{1567}\left(29x_{5n+7}-9573y_{5n+5}+145x_{3n+5}-47865y_{3n+3}+290x_{n+3}-95730y_{n+1}\right)$
- $\frac{1}{28}\left(1595x_{5n+7}-9573y_{5n+6}+7975x_{3n+5}-47865y_{3n+4}+15950x_{n+3}-95730y_{n+2}\right)$
- $\frac{1}{3}[267873x_{5n+7}-28719y_{5n+7}+1339365x_{3n+5}-143595y_{3n+5}+267830x_{n+3}-287190y_{n+3}]$
- $\frac{1}{9}[y_{5n+6}-55y_{5n+5}+5y_{3n+4}-275y_{3n+3}+10y_{n+2}-550y_{n+1}]$
- $\frac{1}{504}[y_{5n+7}-3079y_{5n+5}+5y_{3n+5}-15395y_{3n+3}+10y_{n+3}-30790y_{n+1}]$
- $\frac{1}{9}[55y_{5n+7}-3079y_{5n+6}+275y_{3n+5}-15395y_{3n+4}+55y_{n+3}-30790y_{n+2}]$

REMARKABLE OBSERVATIONS:

1.Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbola which are presented in the Table: 2 below

Table: 2 Hyperbolas

S. No	**Hyperbola**	**(P,Q)**
1.	$81P^2-Q^2=1296$	$P=114x_{n+1}-2x_{n+2}$ $Q=\sqrt{87}\left[2x_{n+2}-110x_{n+1}\right]$
2.	$81P^2-Q^2=4064256$	$P=6382x_{n+1}-2x_{n+3}$ $Q=\sqrt{87}\left[2x_{n+3}-6158x_{n+1}\right]$
3.	$P^2-Q^2=144$	$P=174x_{n+1}-18y_{n+1}$ $Q=\sqrt{87}\left[2y_{n+1}-18x_{n+1}\right]$
4.	$P^2-Q^2=112896$	$P=9570x_{n+1}-18y_{n+2}$ $Q=\sqrt{87}\left[2y_{n+2}-1026x_{n+1}\right]$
5.	$P^2-Q^2=353590416$	$P=535746x_{n+1}-18y_{n+3}$ $Q=\sqrt{87}\left[2y_{n+3}-57438x_{n+1}\right]$

6.	$P^2 - Q^2 = 1296$	$P = 57438x_{n+2} - 1026x_{n+3}$ $Q = \sqrt{87}\left[110x_{n+3} - 6158x_{n+2}\right]$
7.	$P^2 - Q^2 = 112896$	$P = 174x_{n+2} - 1026y_{n+1}$ $Q = \sqrt{87}\left[110y_{n+1} - 18x_{n+2}\right]$
8.	$P^2 - Q^2 = 144$	$P = 9570x_{n+2} - 1026y_{n+2}$ $Q = \sqrt{87}\left[110y_{n+2} - 1026x_{n+2}\right]$
9.	$P^2 - Q^2 = 112896$	$P = 535746x_{n+2} - 1026y_{n+3}$ $Q = \sqrt{87}\left[110y_{n+3} - 57438x_{n+2}\right]$
10.	$P^2 - Q^2 = 353590416$	$P = 174x_{n+3} - 57438y_{n+1}$ $Q = \sqrt{87}\left[6158y_{n+1} - 18x_{n+3}\right]$
11.	$P^2 - Q^2 = 112896$	$P = 9570x_{n+3} - 57438y_{n+2}$ $Q = \sqrt{87}\left[6158y_{n+2} - 1026x_{n+3}\right]$
12.	$P^2 - Q^2 = 144$	$P = 535746x_{n+3} - 57438y_{n+3}$ $Q = \sqrt{87}\left[6158y_{n+3} - 57438x_{n+3}\right]$
13.	$87P^2 - Q^2 = 112752$	$P = 2y_{n+2} - 110y_{n+1}$ $Q = \frac{1}{\sqrt{87}}\left[1026y_{n+1} - 18y_{n+2}\right]$
14.	$87P^2 - Q^2 = 353590272$	$P = 2y_{n+3} - 6158y_{n+1}$ $Q = \frac{1}{\sqrt{87}}\left[57438y_{n+1} - 18y_{n+3}\right]$
15.	$87P^2 - Q^2 = 112752$	$P = 110y_{n+3} - 6158y_{n+2}$ $Q = \frac{1}{\sqrt{87}}\left[57438y_{n+2} - 1026y_{n+3}\right]$

2. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabola which are presented in the Table: 3 below:

Table: 3 Parabolas

S. No	Parabola	(R,Q)
1.	$162R - Q^2 = 1296$	$R = 114x_{2n+2} - 2x_{2n+3} + 4$ $Q = \sqrt{87}\left[2x_{n+2} - 110x_{n+1}\right]$

2.	$9072R - Q^2 = 4064256$	$R = 6382x_{2n+2} - 2x_{2n+4} + 224$ $Q = \sqrt{87}[2x_{n+3} - 6158x_{n+1}]$
3.	$6R - Q^2 = 144$	$R = 174x_{2n+2} - 18y_{2n+2} + 12$ $Q = \sqrt{87}[2y_{n+1} - 18x_{n+1}]$
4.	$168R - Q^2 = 112896$	$R = 9570x_{2n+2} - 18y_{2n+3} + 336$ $Q = \sqrt{87}[2y_{n+2} - 1026x_{n+1}]$
5.	$9402R - Q^2 = 353590416$	$R = 535746x_{2n+2} - 18y_{2n+4} + 18804$ $Q = \sqrt{87}[2y_{n+3} - 57438x_{n+1}]$
6.	$18R - Q^2 = 1296$	$R = 57438x_{2n+3} - 1026x_{2n+4} + 36$ $Q = \sqrt{87}[110x_{n+3} - 6158x_{n+2}]$
7.	$168R - Q^2 = 112896$	$R = 174x_{2n+3} - 1026y_{2n+2} + 336$ $Q = \sqrt{87}[110y_{n+1} - 18x_{n+2}]$
8.	$6R - Q^2 = 144$	$R = 9570x_{2n+3} - 1026y_{2n+3}$ $Q = \sqrt{87}[110y_{n+2} - 1026x_{n+2}]$
9.	$168R - Q^2 = 112896$	$R = 535746x_{2n+3} - 1026y_{2n+4} + 336$ $Q = \sqrt{87}[110y_{n+3} - 57438x_{n+2}]$
10.	$9402R - Q^2 = 353590416$	$R = 174x_{2n+4} - 57438y_{2n+2} + 18804$ $Q = \sqrt{87}[6158y_{n+1} - 18x_{n+3}]$
11.	$168R - Q^2 = 112896$	$R = 9570x_{2n+4} - 57438y_{2n+3} + 336$ $Q = \sqrt{87}[6158y_{n+2} - 1026x_{n+3}]$
12.	$6R - Q^2 = 144$	$P = 535746x_{2n+4} - 57438y_{2n+4} + 12$ $Q = \sqrt{87}[6158y_{n+3} - 57438x_{n+3}]$
13.	$1566R - Q^2 = 112752$	$R = 2y_{2n+3} - 110y_{2n+2} + 36$ $Q = \frac{1}{\sqrt{87}}[1026y_{n+1} - 18y_{n+2}]$
14.	$87696R - Q^2 = 353590272$	$R = 2y_{2n+4} - 6158y_{2n+2} + 2016$ $Q = \frac{1}{\sqrt{87}}[57438y_{n+1} - 18y_{n+3}]$

15.	$1566R - Q^2 = 112752$	$R = 110y_{2n+4} - 6158y_{2n+3} + 36$ $Q = \frac{1}{\sqrt{87}}\left[57438y_{n+2} - 1026y_{n+3}\right]$

Conclusion:

As Negative Pell equations are rich in variety, one may search for integer solutions to other choices of Negative Pell equations.

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