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# On the Positive Pell Equation $y^2 = 80x^2 + 41$

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**ABSTRACT:**

The binary quadratic Diophantine equation represented by the positive pellian $y^2 = 80x^2 + 41$ is analyzed for its non-zero distinct solutions. A few interesting relations among the solutions are given. Further, employing the solutions of the above hyperbola, the solutions of other choices of hyperbolas and parabolas are obtained.

**Keywords:** Binary quadratic, hyperbola, parabola, Pell equation, integral solutions.**2010 mathematics subject classification: 11D09**

## Introduction:

A binary quadratic equation of the form $y^2 = Dx^2 + 1$, where D is non-square positive integer has been studied by various mathematicians for its non-trivial integral solutions when D takes different integral values [1-3]. For an extensive review of various problems, one may refer [4-11]. In this communication, yet another interesting hyperbola given by $y^2 = 80x^2 + 41$is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are obtained. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbola and parabola.

## METHOD OF ANALYSIS:

The Positive Pell equation representing hyperbola under consideration is

$$y^2 = 80x^2 + 41 \qquad (1)$$

whose smallest positive integer solution is

$$x_0 = 1, y_0 = 11$$

To obtain the other solution of (1), consider the pell equation

$$y^2 = 80x^2 + 1$$

whose general solution is given by

$$\tilde{x}_n = \frac{1}{2\sqrt{80}} g_n ; \tilde{y}_n = \frac{1}{2} f_n$$

Where,

$$f_n = \left(9 + \sqrt{80}\right)^{n+1} + \left(9 - \sqrt{80}\right)^{n+1},$$

$$g_n = \left(9 + \sqrt{80}\right)^{n+1} - \left(9 - \sqrt{80}\right)^{n+1}$$

Applying Brahmagupta lemma between $(x_0, y_0)$&$(\tilde{x}_n, \tilde{y}_n)$ ,the other integer solution of (1) are given by

$$x_{n+1} = \frac{1}{2} f_n + \frac{11}{2\sqrt{80}} g_n$$

$$y_{n+1} = \frac{11}{2} f_n + \frac{\sqrt{80}}{2} g_n$$

The recurrence relations satisfied by x and y are given by

$$x_{n+1} - 18x_{n+2} + x_{n+3} = 0$$

$$y_{n+1} - 18y_{n+2} + y_{n+3} = 0$$

A few numerical examples are given in the following table (:1

**Table : 1 Numerical values**

| n | $x_n$ | $y_n$ |
|---|---|---|
| 0 | 1 | 11 |
| 1 | 20 | 179 |
| 2 | 359 | 3211 |
| 3 | 6442 | 57619 |
| 4 | 115597 | 1033931 |

From the above table, we observe some interesting properties among the solutions which are presenting below:

i. $x_n$ values are odd and even.

ii. $y_n$ values are always odd.

**1. Relations between solutions**

- $x_{n+1} - 18x_{n+2} + x_{n+3} = 0$
- $y_{n+1} - x_{n+2} + 9x_{n+1} = 0$
- $y_{n+2} - 9x_{n+2} + x_{n+1} = 0$
- $y_{n+3} - 161x_{n+2} + 9x_{n+1} = 0$
- $18y_{n+1} - x_{n+3} + 161x_{n+1} = 0$
- $2y_{n+2} - x_{n+3} + x_{n+1} = 0$
- $y_{n+2} - 9y_{n+1} - 80x_{n+1} = 0$
- $9y_{n+3} - 161y_{n+2} - 80x_{n+1} = 0$
- $18y_{n+3} - 161x_{n+3} + x_{n+1} = 0$
- $y_{n+3} - 161y_{n+1} - 1440x_{n+1} = 0$
- $y_{n+1} - 9x_{n+3} + 161x_{n+2} = 0$
- $y_{n+2} - x_{n+3} + 9x_{n+2} = 0$
- $y_{n+3} - 9x_{n+3} + x_{n+2} = 0$
- $9y_{n+2} - y_{n+1} - 80x_{n+2} = 0$

- $y_{n+3} - y_{n+1} - 160x_{n+2} = 0$
- $161y_{n+2} - 9y_{n+1} - 80x_{n+3} = 0$
- $161y_{n+3} - y_{n+1} - 1440x_{n+3} = 0$
- $18y_{n+2} - y_{n+3} - y_{n+1} = 0$
- $9y_{n+3} - y_{n+2} + 80x_{n+3} = 0$
- $y_{n+3} - 9y_{n+2} + 80x_{n+2} = 0$

**2. Each of the following expressions represents a Nasty Number**

- $\frac{1}{41}\left(22x_{2n+3} - 358x_{2n+2} + 82\right)$
- $\frac{1}{738}\left(22x_{2n+4} - 6422x_{2n+2} + 1476\right)$
- $\frac{1}{41}\left(22y_{2n+2} - 160x_{2n+2} + 82\right)$
- $\frac{1}{369}\left(22y_{2n+3} - 3200x_{2n+2} + 738\right)$
- $\frac{1}{6601}\left(22y_{2n+4} - 57440x_{2n+2} + 13202\right)$
- $\frac{1}{41}\left(358x_{2n+4} - 6422x_{2n+3} + 82\right)$
- $\frac{1}{41}\left(358y_{2n+3} - 3200x_{2n+3} + 82\right)$
- $\frac{1}{369}\left(358y_{2n+2} - 160x_{2n+3} + 738\right)$
- $\frac{1}{369}\left(358y_{2n+4} - 57440x_{2n+3} + 738\right)$
- $\frac{1}{6601}\left(6422y_{2n+2} - 160x_{2n+4} + 13202\right)$
- $\frac{1}{369}\left(6422y_{2n+3} - 3200x_{2n+4} + 738\right)$
- $\frac{1}{41}\left(6422y_{2n+4} - 57440x_{2n+4} + 82\right)$
- $\frac{1}{41}\left(40y_{2n+2} - 2y_{2n+3} + 82\right)$
- $\frac{1}{738}\left(718y_{2n+2} - 2y_{2n+4} + 1476\right)$
- $\frac{1}{41}\left(718y_{2n+3} - 40y_{2n+4} + 82\right)$

**3. Each of the following expressions represents a Cubical Integer**

- $\frac{1}{41}[22x_{3n+4} - 358x_{3n+3} + 66x_{n+2} - 1074x_{n+1}]$
- $\frac{1}{738}[22x_{3n+5} - 6422x_{3n+3} + 66x_{n+3} - 19266x_{n+1}]$
- $\frac{1}{41}[22y_{3n+3} - 160x_{3n+3} + 66y_{n+1} - 480x_{n+1}]$
- $\frac{1}{369}[22y_{3n+5} - 3200x_{3n+3} + 66y_{n+2} - 9600x_{n+1}]$
- $\frac{1}{6601}[22y_{3n+5} - 57440x_{3n+3} + 66y_{n+3} - 173320x_{n+1}]$
- $\frac{1}{41}[358x_{3n+5} - 6422x_{3n+4} + 1074x_{n+3} - 19266x_{n+2}]$
- $\frac{1}{369}[358y_{3n+3} - 160x_{3n+4} + 1074y_{n+1} - 480x_{n+2}]$
- $\frac{1}{41}[358y_{3n+4} - 3200x_{3n+4} + 1074y_{n+2} - 9600x_{n+2}]$
- $\frac{1}{369}[358y_{3n+5} - 57440x_{3n+4} + 1074y_{n+3} - 172320x_{n+2}]$
- $\frac{1}{6601}[6422y_{3n+3} - 160x_{3n+5} + 19266y_{n+1} - 480x_{n+3}]$
- $\frac{1}{369}[6422y_{3n+4} - 3200x_{3n+5} + 19266y_{n+2} - 9600x_{n+3}]$
- $\frac{1}{41}[6422y_{3n+5} - 57440x_{3n+5} + 19266y_{n+3} - 172320x_{n+3}]$
- $\frac{1}{41}[40y_{3n+3} - 2y_{3n+4} + 120y_{n+1} - 6y_{n+2}]$
- $\frac{1}{738}[718y_{3n+3} - 2y_{3n+5} + 2154y_{n+1} - 6y_{n+3}]$
- $\frac{1}{41}[718y_{3n+4} - 40y_{3n+5} + 2154y_{n+2} - 120y_{n+3}]$

**4. Each of the following expression represents a Bi-quadratic Integer**

- $\frac{1}{41}[22x_{4n+5} - 358x_{4n+4} + 88x_{2n+3} - 1432x_{2n+2} + 246]$
- $\frac{1}{738}[22x_{4n+6} - 6422x_{4n+4} + 88x_{2n+4} - 25688x_{2n+2} + 4428]$
- $\frac{1}{41}[22y_{4n+4} - 160x_{4n+4} + 88y_{2n+2} - 640x_{2n+2} + 246]$
- $\frac{1}{369}[22y_{4n+5} - 3200x_{4n+4} + 88y_{2n+3} - 12800x_{2n+2} + 2214]$
- $\frac{1}{6601}[22y_{4n+6} - 57440x_{4n+4} + 88y_{2n+4} - 229760x_{2n+2} + 39606]$
- $\frac{1}{41}[358x_{4n+6} - 6422x_{4n+5} + 1432x_{2n+4} - 25688x_{2n+3} + 246]$
- $\frac{1}{369}[358y_{4n+4} - 160x_{4n+5} + 1432y_{2n+2} - 640x_{2n+3} + 2214]$

- $\frac{1}{41}[358y_{4n+5} - 3200x_{4n+5} + 1432y_{2n+2} - 12800x_{2n+2} + 246]$
- $\frac{1}{369}[358y_{4n+6} - 57440x_{4n+5} + 1432y_{2n+4} - 229760x_{2n+3} + 2214]$
- $\frac{1}{6601}[6422y_{4n+4} - 160x_{4n+6} + 25688y_{2n+2} - 640x_{2n+4} + 39606]$
- $\frac{1}{369}[6422y_{4n+5} - 3200x_{4n+6} + 25688y_{2n+3} - 12800x_{2n+4} + 2214]$
- $\frac{1}{41}[6422y_{4n+6} - 57440x_{4n+6} + 25688y_{2n+4} - 229760x_{2n+4} + 246]$
- $\frac{1}{41}[40y_{4n+4} - 2y_{4n+5} + 160y_{2n+2} - 8y_{2n+3} + 246]$
- $\frac{1}{738}[718y_{4n+4} - 2y_{4n+6} + 2872y_{2n+2} - 8y_{2n+4} + 4428]$
- $\frac{1}{41}[718y_{4n+5} - 40y_{4n+6} + 2872y_{2n+3} - 160y_{2n+4} + 246]$

**5. Each of the following expressions represents a Quintic Integer**

- $\frac{1}{41}[22x_{5n+6} - 358x_{5n+5} + 110x_{3n+4} - 1790x_{3n+3} + 220x_{n+2} - 8950x_{n+1}]$
- $\frac{1}{738}[22x_{5n+7} - 6422x_{4n+5} + 110x_{3n+5} - 32110x_{3n+3} + 220x_{n+3} - 64190x_{n+1}]$
- $\frac{1}{41}[22y_{5n+5} - 160x_{5n+5} + 110y_{3n+3} - 800x_{3n+3} + 220y_{n+1} - 1600x_{n+1}]$
- $\frac{1}{369}[22y_{5n+6} - 3200x_{5n+5} + 110y_{3n+4} - 16000x_{3n+3} + 220y_{n+2} - 32000x_{n+1}]$
- $\frac{1}{6601}[22y_{5n+7} - 57440x_{5n+5} + 110y_{3n+5} - 287200x_{3n+3} + 220y_{n+3} - 579400x_{n+1}]$
- $\frac{1}{41}[358x_{5n+7} - 6422x_{5n+6} + 1790x_{3n+5} - 32110x_{3n+4} + 3580x_{n+3} - 64220x_{n+2}]$
- $\frac{1}{369}[358y_{5n+5} - 160x_{5n+6} + 1790y_{3n+3} - 800x_{3n+4} + 3580y_{n+1} - 1600x_{n+2}]$
- $\frac{1}{41}[358y_{5n+6} - 3200x_{5n+6} + 1790y_{3n+4} - 16000x_{3n+4} + 3580y_{n+2} - 32000x_{n+2}]$
- $\frac{1}{369}[358y_{5n+7} - 57440x_{5n+6} + 1790y_{3n+5} - 287200x_{3n+4} + 3580y_{n+3} - 574400x_{n+2}]$
- $\frac{1}{6601}[6422y_{5n+5} - 160x_{5n+7} + 32110y_{3n+3} - 800x_{3n+5} + 64220y_{n+1} - 1600x_{n+3}]$
- $\frac{1}{369}[6422y_{5n+6} - 3200x_{5n+7} + 32110y_{3n+4} - 16000x_{3n+5} + 64220y_{n+2} - 32000x_{n+3}]$
- $\frac{1}{41}[6422y_{5n+7} - 57440x_{5n+7} + 32110y_{3n+5} - 287200x_{3n+5} + 64220y_{n+3} - 574400x_{n+3}]$
- $\frac{1}{41}[40y_{5n+5} - 2y_{5n+6} + 200y_{3n+3} - 10y_{3n+4} + 400y_{n+1} - 20y_{n+2}]$
- $\frac{1}{738}[718y_{5n+5} - 2y_{5n+7} + 3590y_{3n+3} - 10y_{3n+5} + 71880y_{n+1} - 20y_{n+3}]$

- $\frac{1}{41}[718y_{5n+6} - 40y_{5n+7} + 3590y_{3n+4} - 200y_{3n+5} + 7180y_{n+2} - 400y_{n+3}]$

**REMARKABLE OBSERVATIONS:**

1. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbola which are presented in the Table: 2 below

**Table: 2 Hyperbola**

| S. No | Hyperbola | (P,Q) |
|---|---|---|
| 1. | $P^2 - Q^2 = 6724$ | $P = 22x_{n+2} - 358x_{n+1}$<br>$Q = \sqrt{80}[40x_{n+1} - 2x_{n+2}]$ |
| 2. | $P^2 - Q^2 = 2178576$ | $P = 22x_{n+3} - 6422x_{n+1}$<br>$Q = \sqrt{80}[718x_{n+1} - 2x_{n+3}]$ |
| 3. | $P^2 - Q^2 = 6724$ | $P = 22y_{n+1} - 160x_{n+1}$<br>$Q = \sqrt{80}[22x_{n+1} - 2y_{n+1}]$ |
| 4. | $P^2 - Q^2 = 544644$ | $P = 22y_{n+2} - 3200x_{n+1}$<br>$Q = \sqrt{80}[358x_{n+1} - 2y_{n+2}]$ |
| 5. | $P^2 - Q^2 = 174292804$ | $P = 22y_{n+3} - 57440x_{n+1}$<br>$Q = \sqrt{80}[6422x_{n+1} - 2y_{n+3}]$ |
| 6. | $P^2 - Q^2 = 6724$ | $P = 358x_{n+3} - 6422x_{n+2}$<br>$Q = \sqrt{80}[718x_{n+2} - 40x_{n+3}]$ |
| 7. | $P^2 - Q^2 = 544644$ | $P = 358y_{n+1} - 160x_{n+2}$<br>$Q = \sqrt{80}[22x_{n+2} - 40y_{n+1}]$ |
| 8. | $P^2 - Q^2 = 6724$ | $P = 358y_{n+2} - 3200x_{n+2}$<br>$Q = \sqrt{80}[358x_{n+2} - 40y_{n+2}]$ |
| 9. | $P^2 - Q^2 = 544644$ | $P = 358y_{n+3} - 57440x_{n+2}$<br>$Q = \sqrt{80}[6422x_{n+2} - 40y_{n+3}]$ |
| 10. | $P^2 - Q^2 = 174292804$ | $P = 6422y_{n+1} - 160x_{n+3}$<br>$Q = \sqrt{80}[22x_{n+3} - 718y_{n+1}]$ |
| 11. | $P^2 - Q^2 = 544644$ | $P = 6422y_{n+2} - 3200x_{n+3}$<br>$Q = \sqrt{80}[358x_{n+3} - 718y_{n+2}]$ |
| 12. | $P^2 - Q^2 = 6724$ | $P = 6422y_{n+3} - 57440x_{n+3}$<br>$Q = \sqrt{80}[6422x_{n+3} - 718y_{n+3}]$ |
| 13. | $P^2 - Q^2 = 6724$ | $P = 40y_{n+1} - 2y_{n+2}$<br>$Q = \frac{1}{\sqrt{80}}[22y_{n+2} - 358y_{n+1}]$ |

| | | |
|---|---|---|
| 14. | $P^2 - Q^2 = 2178576$ | $P = 718y_{n+1} - 2y_{n+3}$ <br> $Q = \frac{1}{\sqrt{80}}[22y_{n+3} - 6422y_{n+1}]$ |
| 15. | $P^2 - Q^2 = 6724$ | $P = 718y_{n+2} - 40y_{n+3}$ <br> $Q = \frac{1}{\sqrt{80}}[358y_{n+3} - 6422y_{n+2}]$ |

2. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabola which are presented in the Table: 3 below:

**Table :3 Parabola**

| S. No | Parabola | (R,Q) |
|---|---|---|
| 1. | $41R - Q^2 = 6724$ | $R = 22x_{2n+3} - 358x_{2n+2} + 82$ <br> $Q = \sqrt{80}[40x_{n+1} - 2x_{n+2}]$ |
| 2. | $738R - Q^2 = 2178576$ | $R = 22x_{2n+4} - 6422x_{2n+2} + 1476$ <br> $Q = \sqrt{80}[718x_{n+1} - 2x_{n+3}]$ |
| 3. | $41R - Q^2 = 6724$ | $R = 22y_{2n+2} - 160x_{2n+2} + 82$ <br> $Q = \sqrt{80}[22x_{n+1} - 2y_{n+1}]$ |
| 4. | $369R - Q^2 = 544644$ | $R = 22y_{2n+3} - 3200x_{2n+2} + 738$ <br> $Q = \sqrt{80}[358x_{n+1} - 2y_{n+2}]$ |
| 5. | $6601R - Q^2 = 174292804$ | $R = 22y_{2n+4} - 57440x_{2n+2} + 13202$ <br> $Q = \sqrt{80}[6422x_{n+1} - 2y_{n+3}]$ |
| 6. | $41R - Q^2 = 6724$ | $R = 358x_{2n+4} - 6422x_{2n+3} + 82$ <br> $Q = \sqrt{80}[718x_{n+2} - 40x_{n+3}]$ |
| 7. | $369R - Q^2 = 544644$ | $R = 358y_{2n+2} - 160x_{2n+3} + 738$ <br> $Q = \sqrt{80}[22x_{n+2} - 40y_{n+1}]$ |
| 8. | $41R - Q^2 = 6724$ | $R = 358y_{2n+3} - 3200x_{2n+3} + 82$ <br> $Q = \sqrt{80}[358x_{n+2} - 40y_{n+2}]$ |
| 9. | $369R - Q^2 = 544644$ | $R = 358y_{2n+4} - 57440x_{2n+3} + 738$ <br> $Q = \sqrt{80}[6422x_{n+2} - 40y_{n+3}]$ |
| 10. | $6601R - Q^2 = 174292804$ | $R = 6422y_{2n+2} - 160x_{2n+4} + 13202$ <br> $Q = \sqrt{80}[22x_{n+3} - 718y_{n+1}]$ |
| 11. | $369R - Q^2 = 544644$ | $R = 6422y_{2n+3} - 3200x_{2n+4} + 738$ <br> $Q = \sqrt{80}[358x_{n+3} - 718y_{n+2}]$ |

| | | |
|---|---|---|
| 12. | $41R - Q^2 = 6724$ | $R = 6422y_{2n+4} - 57440x_{2n+4} + 82$<br>$Q = \sqrt{80}[6422x_{n+3} - 718y_{n+3}]$ |
| 13. | $41R - Q^2 = 6724$ | $R = 40y_{2n+2} - 2y_{2n+3} + 82$<br>$Q = \frac{1}{\sqrt{80}}[22y_{n+2} - 358y_{n+1}]$ |
| 14. | $738R - Q^2 = 2178576$ | $R = 718y_{2n+2} - 2y_{2n+4} + 1476$<br>$Q = \frac{1}{\sqrt{80}}[22y_{n+3} - 6422y_{n+1}]$ |
| 15. | $41R - Q^2 = 6724$ | $R = 718y_{2n+3} - 40y_{2n+4} + 82$<br>$Q = \frac{1}{\sqrt{80}}[358y_{n+3} - 6422y_{n+2}]$ |

## Conclusion:

In this paper, we have presented infinitely many integer solutions for the Diophantine equations represented by the positive pell equation$y^2 = 80x^2 + 41$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of Pell equations and determine their solutions with the suitable properties.

**References:**


David M. Burton, *Elementary Number Theory*, Tata MC Graw hill Publishing company, limited New Delhi -2000.

Mordel, L.J.,1969. *Diophantine equations*. New York: Academic Press.

Telang S.J, Number Theory, Tata Mc Graw Hill publishing, Company limited New Delhi-2000.

Dr .M .A. Gopalan, Dr. A. Kavitha & A. Jesintha Mary-"On the Positive Pell Equation $y^2 = 72x^2 + 36$ ", International Journal of Research- Granthaalayah, ISSN(Print):2394-3629, ISSN(Online): 2350-0530, Vol 5 , Issue 7 , Page No:68-74,July 2017.

Dr.A.Kavitha & S.Ramya-" On the Positive Pellian Equation $y^2 = 90x^2 + 31$", Journal of Mathematics and Informatics,IF:1.627,Vol 10, Page No:41-48,Dec 2017.

Ms.D.Maheswari &R.Suganya-"Observations on the Pell Equation $y^2 = 11x^2 + 5$, International Journal of Emerging Technologies in Engineering Research, IF:4.225,ISSN:Vol 6,Issue 5, Page No:45-50,May 2018.

Dr.T.R.Usharani & K.Dhivya-" On the Positive Pell Equation$y^2 = 14x^2 + 18$", International Journal of Academic Research and Development,ICI,IF:5.22,ISSN:Vol 3, Issue 3, Page No:43-50,,May 2018.

Dr.J.Shanthi, Dr.M.A.Gopalan, "A Study on the positive Pell Equation $y^2 = 42x^2 + 7$", International Research Journal of Education and Technology, ISSN :Vol 01, Issue 05, Page No:107-118, Jan 2021.

Dr.J.Shanthi, Ms.V.Bahavathi, Dr.M.A.Gopalan, "On the Positive Pell Equation$y^2 = 35x^2 + 29$", International Journal of Enhanced Research in Science , Technology and Engineering, Vol 10, Issue 4 ,Page No:40-48, April 2021.

Dr.T.Mahalakshmi, P.Sowmiya, "Observations of the Positive Pell Equation $x^2 = 42y^2 + 28$" May,2023. International Journal of Research Publication and reviews, Volume 4, Page No :5, 570-583, Issue .no:2582-7421.

Dr.T.Mahalakshmi, E.Shalini , "On the Positive Pell Equation $x^2 = 42y^2 + 7$", Jan-2023, International Journal of Research Publication and reviews, Vol-4, Page .No:1 , 763-767, 2582-7421.