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On the Positive Pell Equation $y^2=80x^2+4$

T. Mahalakshmi[1], J. Shanthi[2], M. Meena[3]

[1,2] Assistant Professor, Department of Mathematics, Shrimati Indira Gandhi college, Affiliated to Bharathidasan University, Trichy- 620 002, Tamil Nadu India.

Email: aakashmahalakshmi06@gmail.com Email: shanthivishvaa@gmail.com

[3] M.Sc Scholar, Department of Mathematics, Shrimati Indira Gandhi college, Affiliated to Bharathidasan University, Trichy- 620 002,Tamil Nadu, India.

Email: meena6383895041@gmail.com

ABSTRACT:

The binary quadratic Diophantine equation represented by the positive Pellian $y^2 = 80x^2 + 4$ is analysed for its non-zero distinct solutions. A few interesting relations among the solutions are given. Further, employing the solutions of the above hyperbola, the solutions of other choices of hyperbolas and parabolas are obtained.

Keywords: Binary quadratic, Hyperbola, Parabola, Pell equation, integral solutions. 2010 mathematics subject classification: 11D09

Introduction:

A binary quadratic equation of the form $y^2 = Dx^2 + 1$, where D is non-square positive integer has been studied by various mathematicians for its non-trivial integral solutions when D takes different integral values [1-2]. For an extensive review of various problems, one may refer [3-10]. In this communication, yet another interesting hyperbola given by $y^2 = 80x^2 + 4$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are obtained. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbola, parabola.

METHOD OF ANALYSIS:

The Positive Pell equation representing hyperbola under consideration is

$$y^2 = 80x^2 + 4 \qquad (1)$$

whose smallest positive integer solution is

$$x_0 = 2, y_0 = 18$$

To obtain the other solution of (1), consider the Pell equation

$$y^2 = 80x^2 + 1$$

whose general solution is given by

$$\tilde{x}_n = \frac{1}{2\sqrt{80}} g_n; \tilde{y}_n = \frac{1}{2} f_n$$

where

$$f_n = (9+\sqrt{80})^{n+1} + (9-\sqrt{80})^{n+1}$$

$$g_n = \left(9+\sqrt{80}\right)^{n+1} - \left(9-\sqrt{80}\right)^{n+1}$$

Applying Brahmagupta lemma between (x_0, y_0)&$(\tilde{x}_n, \tilde{y}_n)$ the other integer solution of (1) are given by

$$x_{n+1} = f_n + \frac{9}{\sqrt{80}} g_n$$

$$y_{n+1} = 9 f_n + \frac{80}{\sqrt{80}} g_n$$

The recurrence relations satisfied by x and y are given by

$$x_{n+1} - 18x_{n+2} + x_{n+3} = 0$$

$$y_{n+1} - 18y_{n+2} + y_{n+3} = 0$$

A few numerical examples are given in the following Table:1

Table :1 Numerical examples

N	x_n	y_n
0	2	18
1	36	322
2	646	5778
3	11592	103682
4	208010	1860498

From the above table, we observe some interesting properties among the solutions which are presented below:

x_n & y_n values are even.

1. Relations between solutions

- $x_{n+1} - 18x_{n+2} + x_{n+3} = 0$
- $y_{n+1} - x_{n+2} + 9x_{n+1} = 0$
- $y_{n+2} - 9x_{n+2} + x_{n+1} = 0$
- $y_{n+3} - 161x_{n+2} + 9x_{n+1} = 0$
- $18y_{n+1} - x_{n+3} + 161x_{n+1} = 0$
- $2y_{n+2} - x_{n+3} + x_{n+1} = 0$
- $y_{n+2} - 9y_{n+1} - 80x_{n+1} = 0$
- $9y_{n+3} - 161y_{n+2} - 80x_{n+1} = 0$
- $18y_{n+3} - 161x_{n+3} + x_{n+1} = 0$
- $y_{n+3} - 161y_{n+1} - 1440x_{n+1} = 0$
- $y_{n+1} - 9x_{n+3} + 161x_{n+2} = 0$
- $y_{n+2} - x_{n+3} + 9x_{n+2} = 0$
- $y_{n+3} - 9x_{n+3} + x_{n+2} = 0$

- $9y_{n+2}-y_{n+1}-80x_{n+2}=0$
- $y_{n+3}-y_{n+1}-160x_{n+2}=0$
- $161y_{n+2}-9y_{n+1}-80x_{n+3}=0$
- $161y_{n+3}-y_{n+1}-1440x_{n+3}=0$
- $18y_{n+2}-y_{n+3}-y_{n+1}=0$
- $9y_{n+3}-y_{n+2}+80x_{n+3}=0$
- $y_{n+3}-9y_{n+2}+80x_{n+2}=0$

2. Each of the following expressions represents a Nasty Number

- $\frac{1}{4}\left(36x_{2n+3}-644x_{2n+2}+8\right)$
- $\frac{1}{4}\left(2x_{2n+4}-642x_{2n+2}+8\right)$
- $\frac{1}{4}\left(36y_{2n+3}-320x_{2n+2}+8\right)$
- $\frac{1}{2}\left(2y_{2n+3}-320x_{2n+2}+4\right)$
- $\frac{1}{161}\left(9y_{2n+4}-25840x_{2n+2}+322\right)$
- $\frac{1}{4}\left(644x_{2n+4}-11556x_{2n+3}+8\right)$
- $\frac{1}{36}\left(644x_{2n+4}-320x_{2n+3}+72\right)$
- $\frac{1}{4}\left(644y_{2n+3}-5760x_{2n+3}+8\right)$
- $\frac{1}{9}\left(161y_{2n+4}-25840x_{2n+3}+18\right)$
- $\frac{1}{644}\left(11556y_{2n+2}-320x_{2n+4}+1288\right)$
- $\frac{1}{36}\left(11556y_{2n+3}-5760x_{2n+4}+72\right)$
- $\frac{1}{4}(11556y_{2n+4}-103360x_{3n+4}+8)$
- $\frac{1}{320}\left(5760y_{2n+2}-320y_{2n+3}+640\right)$
- $\frac{1}{5760}\left(103360y_{2n+3}-320y_{2n+4}+11520\right)$
- $\frac{1}{320}\left(103360y_{2n+3}-5760y_{2n+4}+640\right)$

3. Each of the following expressions represents a cubical integer

- $\frac{1}{4}\left[36x_{3n+4}-644x_{3n+3}+108x_{n+2}-1932x_{n+1}\right]$
- $\frac{1}{4}[2x_{3n+5}-642x_{3n+3}+6x_{n+3}-1926x_{n+1}]$
- $\frac{1}{4}[36y_{3n+3}-320x_{3n+3}+108y_{n+1}-960x_{n+1}]$
- $\frac{1}{2}[2y_{3n+4}-320x_{3n+3}+6y_{n+2}-960x_{n+1}]$
- $\frac{1}{161}[9y_{3n+5}-25840x_{3n+3}27y_{n+3}-77520x_{n+1}]$
- $\frac{1}{4}[644x_{3n+5}-11556x_{3n+4}+1932x_{n+3}-34668x_{n+2}]$
- $\frac{1}{36}[644y_{3n+3}-320x_{3n+4}+1932y_{n+1}-960x_{n+2}]$
- $\frac{1}{4}[644y_{3n+4}-5760x_{3n+4}+1932y_{n+2}-1728x_{n+2}]$
- $\frac{1}{9}[161y_{3n+5}-25840x_{3n+4}+483y_{n+3}-77520x_{n+2}]$
- $\frac{1}{161}[2889y_{3n+3}-80x_{3n+5}+8667y_{n+1}-240x_{n+3}$
- $\frac{1}{36}[11556y_{3n+4}-5760x_{3n+5}+34668y_{n+2}-17280x_{n+3}]$
- $\frac{1}{4}[11556y_{3n+5}-103360x_{3n+5}+34668y_{n+3}-310080x_{n+3}]$
- $\frac{1}{320}[5760y_{3n+3}-320y_{3n+4}+17280y_{n+1}-960y_{n+2}]$
- $\frac{1}{8}[323y_{3n+3}-y_{3n+5}+969y_{n+1}-3y_{n+3}]$
- $\frac{1}{320}[103360y_{3n+4}-5760y_{3n+5}+310080y_{n+2}-17280y_{n+3}]$

4. Each of the following expression represents a bi-quadratic integer

- $\frac{1}{4}\left[36x_{4n+5}-644x_{4n+4}+144x_{2n+3}-2576x_{2n+2}+24\right]$
- $\frac{1}{4}[2x_{4n+6}-642x_{4n+4}+8x_{2n+4}-2568x_{2n+2}+24]$
- $\frac{1}{4}[36y_{4n+4}-320x_{4n+4}+144y_{2n+2}-1280x_{2n+2}24]$
- $\frac{1}{2}[2y_{4n+5}-320x_{4n+4}+8y_{2n+3}-1280x_{2n+21}+12]$
- $\frac{1}{161}[9y_{4n+6}-25840x_{4n+4}36y_{2n+4}-103360x_{2n+2}+966]$
- $\frac{1}{4}[644x_{4n+6}-11556x_{4n+5}+2576x_{2n+4}-46224x_{2n+3}+24]$
- $\frac{1}{36}[644y_{4n+4}-320x_{4n+5}+2576y_{2n+2}-1280x_{2n+3}+216]$

- $\frac{1}{4}[644y_{4n+5} - 5760x_{4n+5} + 2576y_{2n+3} - 23040x_{2n+3} 24]$
- $\frac{1}{9}[161y_{4n+6} - 25840x_{4n+5} + 644y_{2n+4} - 103360x_{2n+3} + 54]$
- $\frac{1}{644}[11556y_{4n+4} - 320x_{4n+6} + 46224y_{2n+2} - 1280x_{2n+4} + 3864]$
- $\frac{1}{36}[11556y_{4n+5} - 5760x_{4n+6} + 46224y_{2n+3} - 23040x_{2n+4} + 216]$
- $\frac{1}{4}[11556y_{4n+6} - 103360x_{4n+6} + 46224y_{2n+4} - 413440x_{2n+4} + 24]$
- $\frac{1}{320}[5760y_{4n+4} - 320y_{4n+5} + 23040y_{2n+2} - 1280y_{2n+3} + 1920]$
- $\frac{1}{18}[323y_{4n+4} - y_{4n+6} + 1292y_{2n+2} - 4y_{2n+4} + 108]$
- $\frac{1}{320}[103360y_{4n+5} - 5760y_{4n+6} + 413440y_{2n+3} - 23040y_{2n+4} + 1920]$

4. Each of the following expressions represents a Quintic integer

- $\frac{1}{2}[18x_{5n+6} - 322x_{5n+5} + 90x_{3n+4} - 1610x_{3n+3} + 180x_{n+2} - 3220x_{n+1}]$
- $\frac{1}{4}[2x_{5n+7} - 642x_{5n+5} + 10x_{3n+5} - 3210x_{3n+3} + 20x_{n+3} - 6420x_{n+1}]$
- $\frac{1}{4}[36y_{5n+5} - 320x_{5n+5} + 180y_{3n+3} - 1600x_{3n=3} + 360y_{n+1} - 3200x_{n+1}]$
- $\frac{1}{2}[2y_{5n=6} - 320x_{5n=5} + 10y_{3n+4} - 1600x_{3n+3} + 20y_{n=2} - 3200x_{n+1}]$
- $\frac{1}{161}[9y_{5n+7} - 25840x_{5n+4} + 45y_{3n+5} - 129200x_{3n+3} + 90y_{n+3} - 258400x_{n+1}]$
- $\frac{1}{4}[644y_{5n+6} - 5760x_{5n+6} + 3220y_{3n+4} - 2880x_{3n+4} + 6440y_{n=2} - 83520x_{n+2}]$
- $\frac{1}{9}[161y_{5n+7} - 25840x_{5n+6} + 805y_{3n+5} - 129200x_{3n+4} + 1610y_{n+3} - 258400x_{n+2}]$
- $\frac{1}{4}[644x_{5n+7} - 11556x_{5n+6} + 3220x_{3n+5} - 57780x_{3n+4} + 6440x_{n+3} - 115560x_{n+2}]$
- $\frac{1}{36}[644y_{5n+5} - 320x_{5n+6} + 3200y_{3n+3} - 1600x_{3n+4} + 4940y_{n+1} - 3200x_{n+2}]$
- $\frac{1}{644}[11556y_{5n+5} - 320x_{5n+7} + 57780y_{3n+3} - 640x_{3n+5} + 115560y_{n+1} - 3200x_{n+3}]$
- $\frac{1}{36}[11556y_{5n+6} - 5760x_{5n+7} + 57780y_{3n+4} - 28800x_{3n+5} + 115560y_{n+2} - 57600x_{n+3}]$

- $\frac{1}{4}[11556y_{5n+7} - 103360x_{5n+7} + 57780y_{3n+5} - 516800x_{3n+5} + 115560y_{n+3} - 1033600x_{n+3}]$
- $\frac{1}{320}[5760y_{5n+5} - 320y_{5n+6} + 28800y_{3n+3} - 1600y_{3n+4} + 57600y_{n+1} - 3200y_{n+2}]$
- $\frac{1}{18}[323y_{5n+5} - y_{5n+7} + 1615y_{3n+3} - 5y_{3n+5} + 3230y_{n+1} - 10y_{n+3}]$
- $\frac{1}{320}[103360y_{5n+6} - 5760y_{5n+7} + 516800y_{3n+4} - 28800y_{3n+5} - 103360y_{n+2} - 57600y_{n+3}]$

REMARKABLE OBSERVATIONS:

1.Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbola which are presented in the Table: 2 below

Table: 2 Hyperbolas

S. No	Hyperbola	(P,Q)
1.	$P^2 - 4Q^2 = 64$	$P = 36x_{n+2} - 644x_{n+1}$ $Q = \sqrt{80}[36x_{n+1} - 2x_{n+2}]$
2.	$81P^2 - Q^2 = 5184$	$P = 2x_{n+3} - 642x_{n+1}$ $Q = \sqrt{80}[646x_{n+1} - 2x_{n+3}]$
3.	$P^2 - 4Q^2 = 64$	$P = 36y_{n+1} - 320x_{n+1}$ $Q = \sqrt{80}[18x_{n+1} - 2y_{n+1}]$
4.	$81P^2 - Q^2 = 1296$	$P = 2y_{n+2} - 320x_{n+1}$ $Q = \sqrt{80}[322x_{n+1} - 2y_{n+2}]$
5.	$P^2 - Q^2 = 103684$	$P = 9y_{n+3} - 25840x_{n+1}$ $Q = \sqrt{80}[2889x_{n+1} - y_{n+3}]$
6.	$P^2 - Q^2 = 64$	$P = 644x_{n+3} - 11556x_{n+2}$ $Q = \sqrt{80}[1292x_{n+2} - 72x_{n+3}]$
7.	$P^2 - 324Q^2 = 5184$	$P = 644y_{n+1} - 320x_{n+2}$ $Q = \sqrt{80}[2x_{n+2} - 4y_{n+1}]$
8.	$P^2 - Q^2 = 64$	$P = 644y_{n+2} - 5760x_{n+2}$ $Q = \sqrt{80}[644x_{n+2} - 72y_{n=2}]$
9.	$P^2 - Q^2 = 324$	$P = 161y_{n+3} - 25840x_{n+2}$ $Q = \sqrt{80}[2889x_{n+2} - 18y_{n+3}]$
10.	$P^2 - Q^2 = 103684$	$P = 2889y_{n+1} - 80x_{n+3}$ $Q = \sqrt{80}[9x_{n+3} - 323y_{n=1}]$

11.	$P^2 - Q^2 = 5184$	$P = 11556y_{n+2} - 5760x_{n+3}$ $Q = \sqrt{80}[644x_{n+3} - 1292y_{n+2}]$
12.	$P^2 - Q^2 = 64$	$P = 11556y_{n+3} - 103360x_{n+3}$ $Q = \sqrt{80}[11556x_{n+3} - 1292y_{n+3}]$
13.	$P^2 - Q^2 = 409600$	$P = 5760y_{n+1} - 320y_{n+2}$ $Q = \sqrt{80}[36y_{n=2} - 644y_{n+1}]$
14.	$P^2 - Q^2 = 132710400$	$P = 103360y_{n+1} - 320y_{n+3}$ $Q = \sqrt{80}[36y_{n+3} - 11556y_{n+1}]$
15.	$P^2 - Q^2 = 409600$	$P = 103360y_{n+2} - 5760y_{n+3}$ $Q = \sqrt{80}[644y_{n+3} - 11556y_{n+2}]$

2. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabola which are presented in the Table: 3 below:

Table: 3 Parabolas

S. No	Parabola	(R,Q)
1.	$R - Q^2 = 16$	$R = 36x_{2n+3} - 644x_{2n+2} + 8$ $Q = \sqrt{80}[36x_{n+1} - 2x_{n+2}]$
2.	$324R - Q^2 = 5184$	$R = 2x_{2n+4} - 642x_{2n+2} + 8$ $Q = \sqrt{80}\left[646x_{n+1} - 2x_{n+3}\right]$
3.	$R-Q^2=16$	$R = 36y_{2n+2} - 320x_{2n+2} + 8$ $Q = \sqrt{80}[18x_{n+1} - 2y_{n+1}]$
4.	$162R-Q^2=1296$	$R = 2y_{2n+3} - 320x_{2n+2} + 4$ $Q = \sqrt{80}\left[322x_{n+1} - 2y_{n+2}\right]$
5.	$161R-Q^2=103684$	$R = 9y_{2n+4} - 25840x_{2n+2} + 322$ $Q = \sqrt{80}[2889x_{n+1} - y_{n+3}]$
6.	$4R - Q^2 = 64$	$R = 644x_{2n+4} - 11556x_{2n+3} + 8$ $Q = \sqrt{80}[1292x_{n+2} - 72x_{n+3}]$
7.	$R - 9Q^2 = 144$	$R = 644y_{2n+2} - 320x_{2n+3} + 72$ $Q = \sqrt{80}[2x_{n+2} - 4y_{n+1}]$
8.	$4R - Q^2 = 64$	$R = 644y_{2n+3} - 5760x_{2n+3} + 8$ $Q = \sqrt{80}[644x_{n+2} - 72y_{n+2}]$
9.	$9R - Q^2 = 324$	$R = 161y_{2n+4} - 25840x_{2n+3} + 18$ $Q = \sqrt{80}[2889x_{n+2} - 18y_{n+3}]$
10.	$644R - 4Q^2 = 1658944$	$R = 11556y_{2n+2} - 320x_{2n+4} + 1288$ $Q = 4\sqrt{80}[9x_{n+3} - 323y_{n+1}]$

11.	$36R - Q^2 = 5184$	$R = 11556y_{2n+3} - 5760x_{2n+4} + 72$ $Q = \sqrt{80}[644x_{n+3} - 1292y_{n+1}]$
12.	$4R - Q^2 = 64$	$R = 11556y_{2n+4} - 103360x_{2n+4} + 8$ $Q = \sqrt{80}[11556x_{n+3} - 1292y_{n+3}]$
13.	$320R - Q^2 = 409600$	$R = 5760y_{2n+2} - 320y_{2n+3} + 640$ $Q = \sqrt{80}[36y_{n+2} - 644y_{n+1}]$
14.	$5760R - Q^2 = 132710400$	$R = 103360y_{2n+2} - 320y_{2n+4} + 11520$ $Q = \sqrt{80}[36y_{n+3} - 11556y_{n+1}]$
15.	$320R - Q^2 = 409600$	$R = 103360y_{2n+3} - 5760y_{2n+4} + 640$ $Q = \sqrt{80}[644y_{n+3} - 11556y_{n+2}]$

Conclusion:

As positive pell equations are rich in variety, one may search for integer solutions to other choices of positive pell equations.

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