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On the Negative Pell Equation $y^2 = 80x^2 - 31$

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Abstract:

The hyperbola $y^2 = 80x^2 - 31$is studied for its different solutions in integers. Some remarkable relations among the solutions are given. Also, integer solutions for other choices of Hyperbolas and Parabolas based on given solutions of the hyperbola under consideration are exhibited.

Keywords: Binary quadratic, Hyperbola, Parabola, Pell equation, integral solutions. 2010 mathematics subject classification: 11D09

Introduction:

A binary quadratic equation of the form $y^2 = Dx^2 + 1$, where D is non-square positive integer has been studied by various mathematicians for its non-trivial integral solutions when D takes different integral values [1-2]. For an extensive review of various problems, one may refer [3-10]. In this communication, yet another interesting hyperbola given by $y^2 = 80x^2 - 31$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are obtained. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbola, parabola.

Method of Analysis:

The Negative Pell equation representing hyperbola under consideration is

$$y^2 = 80x^2 - 31 \qquad (1)$$

whose smallest positive integer solution is

$$x_0 = 1, y_0 = 7$$

To obtain the other solution of (1), consider the Pell equation

$$y^2 = 80x^2 + 1$$

whose general solution is given by

$$\tilde{x}_n = \frac{1}{2\sqrt{80}} g_n; \tilde{y}_n = \frac{1}{2} f_n$$

where

$$f_n = \left(9 + \sqrt{80}\right)^{n+1} + \left(9 - \sqrt{80}\right)^{n+1},$$

$$g_n = \left(9+\sqrt{80}\right)^{n+1} - \left(9-\sqrt{80}\right)^{n+1}$$

Applying Brahmagupta lemma between (x_0, y_0)&$(\tilde{x}_n, \tilde{y}_n)$ the other integer solution of (1) are given by

$$x_{n+1} = \frac{1}{2} f_n + \frac{7}{2\sqrt{80}} g_n$$

$$y_{n+1} = \frac{7}{2} f_n + \frac{80}{2\sqrt{80}} g_n$$

The recurrence relations satisfied by x and y are given by

$$x_{n+3} - 18x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 18y_{n+2} + y_{n+1} = 0$$

A few numerical examples are given in the following Table:1

Table :1 Numerical examples

n	x_n	y_n
0	1	7
1	16	143
2	287	2567
3	5150	46063
4	92413	826567

From the above table, we observe some interesting properties among the solutions which are presented below:

x_n values are odd and even, whereas y_n values always are odd.

1. Relations between solutions

- $x_{n+3} - 18x_{n+2} + x_{n+1} = 0$
- $y_{n+1} - x_{n+2} + 9x_{n+1} = 0$
- $y_{n+2} - 9x_{n+2} + x_{n+1} = 0$
- $y_{n+3} - 161x_{n+2} + 9x_{n+1} = 0$
- $x_{n+3} - 161x_{n+1} - 18y_{n+1} = 0$

- $y_{n+2} - 80x_{n+1} - 9y_{n+1} = 0$
- $y_{n+3} - 1440x_{n+1} - 161y_{n+1} = 0$
- $x_{n+3} - x_{n+1} - 2y_{n+2} = 0$
- $9y_{n+3} - 80x_{n+1} - 161y_{n+2} = 0$
- $161x_{n+3} - x_{n+1} - 18y_{n+3} = 0$
- $y_{n+2} - x_{n+3} - 9x_{n+2} = 0$
- $y_{n+3} - 80x_{n+2} - 9y_{n+2} = 0$
- $y_{n+1} - y_{n+3} + 160x_{n+2} = 0$
- $9x_{n+3} - x_{n+2} - y_{n+3} = 0$
- $9x_{n+3} - 161x_{n+2} - y_{n+1} = 0$
- $9y_{n+2} - 80x_{n+2} - y_{n+1} = 0$
- $161y_{n+2} - 80x_{n+3} - 9y_{n+1} = 0$
- $161y_{n+3} - 1440x_{n+3} - y_{n+1} = 0$
- $x_{n+2} - 9x_{n+3} + y_{n+3} = 0$
- $y_{n+3} - 18y_{n+2} + y_{n+1} = 0$

2. Each of the following expressions represents a Nasty Number

- $\frac{1}{31}\left(286x_{2n+2} - 14x_{2n+3} + 62\right)$
- $\frac{1}{31}\left(160x_{2n+2} - 14y_{2n+2} + 62\right)$
- $\frac{1}{279}\left(2560x_{2n+2} - 14y_{2n+3} + 558\right)$
- $\frac{1}{713}\left(6560x_{2n+2} - 2y_{2n+4} + 1426\right)$

- $\frac{1}{31}\left(5134x_{2n+3}-286x_{2n+4}+62\right)$
- $\frac{1}{31}\left(2560x_{2n+3}-286y_{2n+3}+62\right)$
- $\frac{1}{279}\left(45920x_{2n+3}-286y_{2n+4}+558\right)$
- $\frac{1}{279}\left(160x_{2n+3}-286y_{2n+2}+558\right)$
- $\frac{1}{4991}\left(160x_{2n+4}-5134y_{2n+2}+9982\right)$
- $\frac{1}{279}\left(2560x_{2n+4}-5134y_{2n+3}+558\right)$
- $\frac{1}{31}\left(45920x_{2n+4}-5134y_{2n+4}+62\right)$
- $\frac{1}{31}(2y_{2n+3}-32y_{3n+2}+62)$
- $\frac{1}{279}\left(2y_{2n+4}-274y_{2n+2}+1116\right)$
- $\frac{1}{279}\left(2567x_{2n+2}-7x_{2n+4}+558\right)$
- $\frac{1}{31}\left(32y_{2n+4}-574y_{2n+3}+187\right)$

3. Each of the following expressions represents a cubical integer

- $\frac{1}{31}[286x_{3n+3}-14x_{3n+4}+858x_{n+1}-42x_{n+2}]$
- $\frac{1}{31}[160x_{3n+3}-14y_{3n+3}+480x_{n+1}-42y_{n+1}]$
- $\frac{1}{279}[2560x_{3n+3}-14y_{3n+4}+7680x_{n+1}-42y_{n+2}]$
- $\frac{1}{713}[6560x_{3n+3}-2y_{3n+5}+19680x_{n+1}-6y_{n+3}]$
- $\frac{1}{31}[5134x_{3n+4}-286x_{3n+5}+16402x_{n+2}-858x_{n+3}]$
- $\frac{1}{31}[2560x_{3n+4}-286y_{3n+4}+7680x_{n+2}-858y_{n+2}]$

- $\frac{1}{279}[45920x_{3n+4} - 286y_{3n+5} + 137760x_{n+2} - 858y_{n+3}]$
- $\frac{1}{279}[160x_{3n+4} - 286y_{3n+3} + 480x_{n+2} - 858y_{n+1}]$
- $\frac{1}{4991}[160x_{3n+5} - 5134y_{3n+3} + 480x_{n+3} - 15402y_{n+1}]$
- $\frac{1}{279}[2560x_{3n+5} - 5134y_{3n+4} + 7680x_{n+3} - 15402y_{n+2}]$
- $\frac{1}{31}[45920x_{3n+5} - 5134y_{3n+5} + 137760x_{n+3} - 15402y_{n+3}]$
- $\frac{1}{31}[2y_{3n+4} - 32y_{3n+3} + 6y_{n+2} - 96y_{n+1}]$
- $\frac{1}{558}[2y_{3n+5} - 574y_{3n+3} + 6y_{n+3} - 1722y_{n+1}]$
- $\frac{1}{279}[2567x_{3n+3} - 7x_{3n+5} + 7701x_{n+1} - 21x_{n+3}]$
- $\frac{1}{31}[32y_{3n+5} - 574y_{3n+4} + 96y_{n+3} - 1722y_{n+2}]$

4. Each of the following expression represents a Bi-quadratic Integer

- $\frac{1}{31}[286x_{4n+4} - 14x_{4n+5} + 1144x_{2n+2} - 56x_{2n+3} + 186]$
- $\frac{1}{31}[160x_{4n+4} - 14y_{4n+4} + 640x_{2n+2} - 56y_{2n+2} + 186]$
- $\frac{1}{279}[2560x_{4n+4} - 14y_{4n+5} + 10240x_{2n+2} - 56y_{2n+3} + 1674]$
- $\frac{1}{713}[6560x_{4n+4} - 2y_{4n+6} + 25240x_{2n+2} - 8y_{2n+4} + 4278]$
- $\frac{1}{31}[5134x_{4n+5} - 286x_{4n+6} + 20536x_{2n+3} - 1144x_{2n+4} + 186]$
- $\frac{1}{31}[2560x_{4n+5} - 286y_{4n+5} + 10240x_{2n+3} - 1144y_{2n+3} + 186]$

- $\frac{1}{279}[45920x_{4n+5}-286y_{4n+6}+183680x_{2n+2}-1144y_{2n+4}+1674]$
- $\frac{1}{279}[160x_{4n+5}-286x_{4n+4}+640x_{2n+3}-1144y_{2n+2}+1674]$
- $\frac{1}{4991}[160x_{4n+6}-5134y_{4n+4}+640x_{2n+4}-20536y_{2n+2}+29946]$
- $\frac{1}{279}[2560x_{4n+6}-5134y_{4n+5}+10240x_{2n+4}-20536y_{2n+3}+1674]$
- $\frac{1}{31}[45920x_{4n+6}-5134y_{4n+6}+183680x_{2n+4}-20536y_{2n+4}+186]$
- $\frac{1}{31}[2y_{4n+5}-32y_{4n+4}-8y_{2n+3}-128y_{2n+2}+186]$
- $\frac{1}{558}[2x_{4n+6}-574y_{4n+4}+8y_{2n+4}-2296y_{2n+2}+3348]$
- $\frac{1}{279}[2567x_{4n+4}-7x_{4n+6}+10268x_{2n+2}-28x_{2n+4}+1674]$
- $\frac{1}{31}[32y_{4n+6}-574y_{4n+5}+128y_{2n+4}-2296y_{2n+3}+186]$

5. Each of the following expressions represents a Quintic integer

- $\frac{1}{31}[286x_{5n+5}-14x_{5n+6}+1430x_{3n+3}-70x_{3n+4}+2860x_{n+1}-140x_{n+2}]$
- $\frac{1}{31}[160x_{5n+5}-14x_{5n+5}+800x_{3n+3}-70y_{3n+3}+1600x_{n+1}-140y_{n+1}]$
- $\frac{1}{279}[2560x_{5n+5}-14y_{5n+6}+12800x_{3n+3}-70y_{3n+4}+25600x_{n+1}-140y_{n+2}]$
- $\frac{1}{713}[6560x_{5n+5}-2y_{5n+7}+32800x_{3n+3}-10x_{3n+5}+65600x_{n+1}-20y_{n+3}]$
- $\frac{1}{31}[5134x_{5n+6}-286x_{5n+7}+25670x_{3n+4}-1430x_{3n+5}+51340x_{n+2}-2860x_{n+2}]$
- $\frac{1}{31}[2560x_{5n+6}-286y_{5n+6}+12800x_{3n+4}-1430y_{3n+4}+25600x_{n+2}-2860y_{n+2}]$
- $\frac{1}{279}[45920x_{5n+6}-286x_{5n+7}+229600x_{3n+4}-1430y_{3n+5}+459200x_{n+2}-2860y_{n+3}]$

- $\frac{1}{279}[160x_{5n+6} - 286y_{5n+5} + 800x_{3n+4} - 1430y_{3n+3} + 1600x_{n+2} - 2860y_{n+1}]$
- $\frac{1}{4991}[160x_{5n+7} - 5134y_{5n+5} + 800x_{3n+5} - 25670y_{3n+3} + 1600x_{n+3} - 51340y_{n+1}]$
- $\frac{1}{279}[2560x_{5n+7} - 5134y_{5n+6} + 12800x_{3n+5} - 25670y_{3n+4} + 25600x_{n+3} - 5134y_{n+2}]$
- $\frac{1}{31}[45920x_{5n+7} - 5134y_{5n+7} + 229600_{3n+5} - 25670y_{3n+5} + 459200x_{n+3} - 51340y_{n+3}]$
- $\frac{1}{31}[2y_{5n+6} - 32y_{5n+5} + 10y_{3n+4} - 160y_{3n+3} + 20y_{n+2} - 320y_{n+1}]$
- $\frac{1}{558}[2x_{5n+7} - 574y_{5n+5} + 10y_{3n+5} - 2870y_{3n+3} + 20y_{n+3} - 5740y_{n+1}]$
- $\frac{1}{279}[2567x_{5n+5} - 7x_{5n+7} + 12835x_{3n+3} - 35x_{3n+5} + 2567x_{n+1} - 70x_{n+3}]$
- $\frac{1}{31}[32y_{5n+7} - 574y_{5n+6} + 160y_{3n+5} - 2870y_{3n+4} + 320y_{n+3} - 5740y_{n+2}]$

Remarkable Observations:

1. .Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbola which are presented in the Table: 2 below

Table: 2 Hyperbolas

S. No	Hyperbola	(P,Q)
1.	$P^2 - Q^2 = 3844$	$P = 286x_{n+1} - 14x_{n+2}$ $Q = \sqrt{80}[2x_{n+2} - 32x_{n+1}]$
2.	$P^2 - Q^2 = 3844$	$P = 160x_{n+1} - 14y_{n+1}$ $Q = \sqrt{80}[2y_{n+1} - 14x_{n+1}]$
3.	$P^2 - Q^2 = 311364$	$P = 2560x_{n+1} - 14y_{n+2}$ $Q = \sqrt{80}[2y_{n+2} - 286x_{n+1}]$
4.	$49P^2 - Q^2 = 99640324$	$P = 6560x_{n+1} - 2y_{n+3}$ $Q = \sqrt{80}[2y_{n+3} - 5134x_{n+1}]$
5.	$P^2 - Q^2 = 3844$	$P = 5134x_{n+2} - 286x_{n+3}$ $Q = \sqrt{80}[32x_{n+3} - 574x_{n+2}]$

6.	$P^2 - Q^2 = 3844$	$P = 2560x_{n+2} - 286y_{n+2}$ $Q = \sqrt{80}[32y_{n+2} - 286x_{n+2}]$
7.	$P^2 - Q^2 = 311364$	$P = 45920x_{n+2} - 286y_{n+3}$ $Q = \sqrt{80}[32y_{n+3} - 5134x_{n+2}]$
8.	$P^2 - Q^2 = 311364$	$P = 160x_{n+2} - 286y_{n+1}$ $Q = \sqrt{80}[32y_{n+1} - 14x_{n+2}]$
9.	$P^2 - Q^2 = 99640324$	$P = 160x_{n+3} - 5134y_{n+1}$ $Q = \sqrt{80}[574y_{n+1} - 14x_{n+3}]$
10.	$P^2 - Q^2 = 311364$	$P = 2560x_{n+3} - 5134y_{n+2}$ $Q = \sqrt{80}[574y_{n+2} - 286x_{n+3}]$
11.	$P^2 - Q^2 = 3844$	$P = 45920x_{n+3} - 5134y_{n+3}$ $Q = \sqrt{80}[574y_{n+3} - 5134x_{n+3}]$
12.	$P^2 - Q^2 = 24601600$	$P = 160y_{n+2} - 2560y_{n+1}$ $Q = \sqrt{80}[286y_{n+1} - 14y_{n+2}]$
13.	$P^2 - Q^2 = 797091840($	$P = 160y_{n+3} - 45920y_{n+1}$ $Q = \sqrt{80}[5134y_{n+1} - 14y_{n+3}]$
14.	$P^2 - Q^2 = 1245456$	$P = 5134x_{n+1} - 14x_{n+3}$ $Q = \sqrt{80}[2x_{n+3} - 574x_{n+1}]$
15.	$P^2 - Q^2 = 24601600$	$P = 2560y_{n+3} - 45920y_{n+2}$ $Q = \sqrt{80}[5134y_{n+2} - 286y_{n+3}]$

2. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabola which are presented in the Table: 3 below:

Table :3 Parabolas

S. No	**Parabola**	**(R,Q)**
1.	$31R - Q^2 = 3844$	$R = 286x_{2n+2} - 14x_{2n+3} + 62$ $Q = \sqrt{80}[2x_{n+2} - 32x_{n+1}]$

2.	$31R - Q^2 = 3844$	$R = 160x_{2n+2} - 14y_{2n+2} + 62$ $Q = \sqrt{80}[2y_{n+1} - 14x_{n+1}]$
3.	$279R - Q^2 = 311364$	$R = 2560x_{2n+2} - 14y_{2n+3} + 558$ $Q = \sqrt{80}[2y_{n+2} - 286x_{n+1}]$
4.	$34937R - Q^2 = 99640324$	$R = 6560x_{2n+2} - 2y_{2n+4} + 1426$ $Q = \sqrt{80}[2y_{n+3} - 5134x_{n+1}]$
5.	$31R - Q^2 = 3844$	$R = 5134x_{2n+3} - 286x_{2n+4} + 62$ $Q = \sqrt{80}[32x_{n+3} - 574x_{n+2}]$
6.	$31R - Q^2 = 3844$	$R = 2560x_{2n+3} - 286y_{2n+3} + 62$ $Q = \sqrt{80}[32y_{n+2} - 286x_{n+2}]$
7.	$279R - Q^2 = 311364$	$R = 45920x_{2n+3} - 286y_{2n+4} + 558$ $Q = \sqrt{80}[32y_{n+3} - 5134x_{n+2}]$
8.	$279R - Q^2 = 311364$	$R = 160x_{2n+3} - 286y_{2n+2} + 558$ $Q = \sqrt{80}[32y_{n+1} - 14x_{n+2}]$
9.	$4991R - Q^2 = 99640324$	$R = 160x_{2n+4} - 5134y_{2n+2} + 9982$ $Q = \sqrt{80}[574y_{n+1} - 14x_{n+3}]$
10.	$279R - Q^2 = 311364$	$R = 2560x_{2n+4} - 5134y_{2n+3} + 558$ $Q = \sqrt{80}[574y_{n+2} - 286x_{n+3}]$
11.	$31R - Q^2 = 3844$	$R = 45920x_{2n+4} - 5134y_{2n+4} + 62$ $Q = \sqrt{80}[574y_{n+3} - 5134x_{n+3}]$
12.	$2480R - Q^2 = 24601600$	$R = 160y_{2n+3} - 2560y_{2n+2} + 4960$ $Q = \sqrt{80}[286y_{n+1} - 14y_{n+2}]$
13.	$44640R - Q^2 = 797091840$(	$R = 160y_{2n+4} - 45920y_{2n+2} + 89280$ $Q = \sqrt{80}[5134y_{n+1} - 14y_{n+3}]$
14.	$558R - Q^2 = 1245456$	$R = 5134x_{2n+2} - 14x_{2n+4} + 1116$ $Q = \sqrt{80}[2x_{n+3} - 574x_{n+1}]$

15.	$2480R - Q^2 = 24601600$	$R = 2560y_{2n+4} - 45920y_{2n+3} + 4960$ $Q = \sqrt{80}[5134y_{n+2} - 286y_{n+3}]$

Conclusion:

As negative pell equations are rich in variety, one may search for integer solutions to other choices of negative pell equations.

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