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Low Dimensional Topology and Braid Group

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ABSTRACT

Low-dimensional topology is a mathematical field that encompasses the exploration of braid groups. These interconnected areas of study offer profound insights into the fundamental characteristics of spaces and the intricate structures that can emerge within them. The investigation of low-dimensional topology extends its reach into various scientific disciplines, making it a dynamic and vital realm of mathematical research that continues to thrive.

Keywords of low dimensional Topology And Braid group

2-Manifolds, 3-Manifolds, Knot Theory, Dehn Surgery, Homeomorphic, Braid Groups, Algebraic Structures, Knot and braid, Represent of Braid group

1. Introduction

Topology, as a branch of mathematics, is dedicated to the examination of spatial properties that remain unaltered under continuous transformations. One specific and captivating subfield of topology is low-dimensional topology, which concentrates on spaces of dimensions two, three, and four. Within the realm of low-dimensional topology, there is a pronounced focus on manifolds, which are spaces that exhibit local characteristics resembling Euclidean space. Manifolds can be methodically classified and studied through various techniques, including differential topology and algebraic topology.

Another intriguing facet of low-dimensional topology is knot theory, which is an intricate study of the properties and classifications of knots and links within three-dimensional space. Knot theory boasts a wide range of applications across diverse fields, including DNA research and physics. Furthermore, braid groups, which are mathematical groups arising from the examination of braids, hold a pivotal role in low-dimensional topology. These groups offer valuable insights into the intricate structure and behaviour of knots and links. In essence, low-dimensional topology, encompassing manifolds, knot theory, and braid groups, offers a captivating and profound realm of exploration within the field of mathematics.

2. Two-dimensional Manifold

A two-dimensional manifold is a topological space that locally resembles Euclidean two-dimensional space (commonly known as the plane). In other words, a two-dimensional manifold is a space where every point has a neighbourhood that is homomorphic to a region in the plane. Formally, a topological space M is considered a two-dimensional manifold if, for every point x in M, there exists an open neighbourhood U(x)and a homeomorphism (a continuous bijection with a continuous inverse) between U and an open subset of the Euclidean plane R^2. Two-dimensional manifolds are often referred to as surfaces, and they come in various forms, including spheres, tori, projective planes, and more complex surfaces.

Definition

The open disk, denoted as D, which consists of all points in R^2 (the two-dimensional Euclidean space) with a distance less than one from the origin, can be shown to be homeomorphic to R^2. This homeomorphism can be established using a specific function, denoted as f, which maps points from the open disk to points in R^2.

The homeomorphism function f is defined as follows: $f(x) = \frac{x}{1-||x||}$ where ||x|| represents the Euclidean norm or the distance of point x from the origin.

It is noteworthy that this homeomorphism demonstrates that every open disk, regardless of its size or location within R^2, can be considered homomorphic to the entire plane R^2. This result is a fundamental concept in topology, highlighting the remarkable flexibility and uniformity of topological spaces.

2.1 Orientability

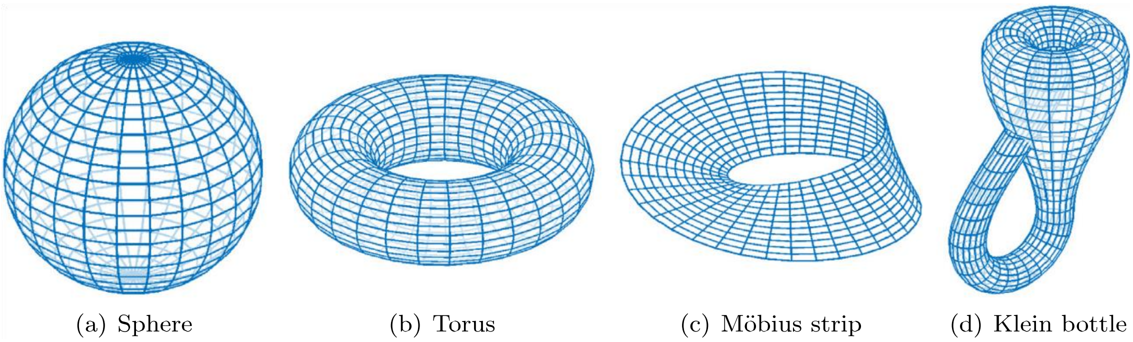


Fig 2.1: Example of 2 Dimensional Manifold

Of the examples we have seen so far, the Mo¨bius strip has the curious property that it seems to have two sides locally at every interior point but there is only one side globally. To express this property intrinsically, without reference to the embedding in R^3, we consider a small, oriented circle inside the strip. We move it around without altering its orientation, like a clock whose fingers keep turning in the same direction. However, if we slide the clock once around the strip its orientation is the reverse of what it used to be and we call the path of its center an orientation-reversing closed curve. There are also

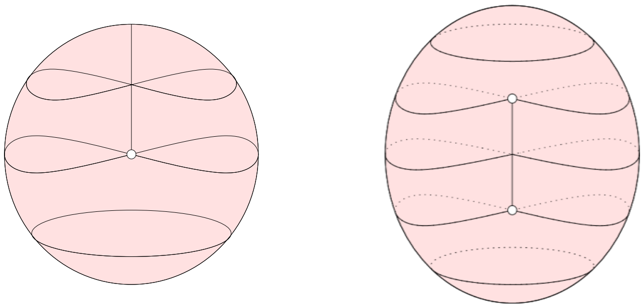


Fig 2.2 Mobius strip

The projective plane, P^2, obtained by gluing a disk to a Mo¨bius strip. Right: the Klein bottle, K^2, obtained by gluing two Mo¨bius strips together. The vertical lines are self-intersections that are forced by placing the 2-manifolds in R^3. They are topologically not important.

orientation-preserving closed curves in the Mo¨bius strip, such as the one that goes around the strip twice. If all closed curves in a 2-manifold are orientation preserving then the 2-manifold is orientable, else it is non-orientable.

Note that the boundary of the Mo¨bius strip is a single circle. We can therefore glue the strip to a sphere or a torus after removing an open disk from the latter. This operation is often referred to as adding a cross-cap. In the first case, we get the projective plane, the sphere with one cross-cap, and in the second case, we get the Klein bottle, the sphere with two cross-caps. Both cannot be embedded in R^3, so we have to draw them with self-intersections, but these should be ignored when we think about these surfaces.

Proposition 2.1.1

Two closed surfaces are homeomorphic if and only if they are both orientable or both nonorientable, and they have the same genus.

We note the well-known fact that the connected sum of T^2 with RP^2 is homeomorphic with the connected sum of three copies of RP^2, both being non orientable of genus 3. This surface is known as Dyck's surface.

The fundamental groups of closed surfaces have presentations: If Σ is the orientable surface of genus g: $\pi_1(\Sigma)$ ~= ha_1,b_1,...,a_g,b_g| [a_1,b_1][a_2,b_2]···[a_g,b_g] = 1i If Σ is the nonorientable surface of genus g:

$$\pi_1(\Sigma) \cong \langle a_1, \ldots, a_g |\ a_1^2 a_2^2 \cdots a_g^2 = 1\rangle$$

Theorem 2.0.1

If Σ is a surface, then $\pi_1(\Sigma)$ is bi-orderable, with two exceptions: the Klein bottle, whose group is only left-orderable and the projective plane which is not left-orderable because it is a torsion group of order 2.

Proof:

First consider the Klein bottle group, which has the alternative presentation hx,y|$y^{-1}xy = x^{-1}$i. If one kills the infinite cyclic subgroup generated by x the resulting quotient is also infinite cyclic. Thus the Klein bottle group is in the middle of a short exact sequence flanked by orderable groups, and is therefore left orderable. It cannot be bi-ordered because the defining relation would then imply the contradiction that x is greater than the identity if and only x^{-1} is greater than the identity.

We now sketch the proof that closed surfaces other than the Klein bottle and RP^2 may be bi-ordered. The proof boils down to the single case of the non orientable surface Σ of genus 3, known as Dyck's surface, by the trick of considering covering spaces. Recall that the projection map of a covering space induces injective homomorphisms of the corresponding fundamental groups. Now non orientable surfaces of genus $g \geq 3$ may be realized as the connected sum of the torus T^2 with $g - 2$ copies of RP^2. In other words, one may remove $g - 2$ disjoint disks from T^2 and replace them by M¨obius bands to construct such a surface. By considering a finite covering of T^2 by itself, and lifting a disk downstairs to multiple disks upstairs, we can construct finite sheeted covers of Σ by all higher genus nonorientable surfaces. Thus their fundamental groups can be considered as (finite index) subgroups of $\pi_1(\Sigma)$. As for the orientable surfaces, we consider the oriented double cover of a nonorientable surface of genus $g \geq 3$. Since Euler characteristics double, we see that the oriented surface upstairs in the cover will have genus $g - 1$. Therefore $\pi_1(\Sigma)$ also contains subgroups isomorphic to the fundamental groups of orientable surfaces of genus 2 or more. This leaves the torus to consider, but its group is just Z^2 which is obviously bi-orderable.

It remains to order $\pi_1(\Sigma)$, where Σ is Dyck's surface, which we can also do by regarding covering spaces. Take the universal cover of T^2 by R^2, and choose a small disk in T^2, which lifts to infinitely many disks in R^2, which we may take centered at the integral points $(m,n) \in R^2$. Now remove all the disks downstairs and upstairs and replace them by M¨obius bands. This produces an infinite-sheeted covering $\Sigma\tilde{} \to \Sigma$. One calculates that $\pi_1(\Sigma)\tilde{}$ is an infinitely-generated free group F_∞ with generators represented by the central curves of the M´obius bands that were sewn to the punctured R^2, connected by tails to some fixed basepoint. Thus we have an exact sequence

$$1 \to F_\infty \to \pi_1(\Sigma) \to Z^2 \to 1$$

in which $\pi_1(\Sigma)$ is sandwiched between bi-ordered groups. The free group F_∞ may be ordered, for example using a Magnus expansion, in such a way that the ordering is invariant under conjugation by elements of $\pi_1(\Sigma)$, that is, deck transformations of the covering. (see [42] for details). We then appeal to proposition 2.1.1.

Note that this corrects a statement in the literature [29], p. 201 "... the fundamental group of a one-sided surface cannot be ordered."

3. Three-Dimensional Manifolds

Many, although not all, of the fundamental groups associated with 3-manifolds exhibit the property of left-orderability, and in some cases, they can even be locally indicable. The orderability of these groups plays a significant role in our understanding of various aspects of 3-manifold theory, such as foliations, mappings, and other related structures. A valuable tool in determining the left-orderability of these groups is the theorem established by Burns and Hale [8], which simplifies the inquiry by reducing it to a local analysis.

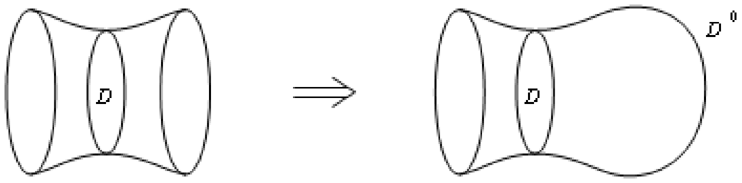


Fig 3: Example of a 3-dimensional Monifold

Foliations 3.1

Let us now consider (codimension one) foliations of 3-manifolds M. By this we mean a collection F of subsets of M for which appropriate R^3 charts at points of M meet members of F in parallel planes in R^3. Members of F are called leaves: they may be closed surfaces, or they may be noncompact and wrap around and meet the chart infinitely many times. It is known that every closed 3-manifold admits such foliations. F is said to be transversely oriented if there is a continuous assignment of normal vectors to all the leaves. If each member of F is considered a point, with the natural decomposition space topology, one gets the "space of leaves" which may be a non-Hausdorff space.

If $M\tilde{} \to M$ is a covering space, then a foliation F of M naturally lifts to a foliation $F\tilde{}$ of $M\tilde{}$. An R-covered foliation F of M is one which, when lifted to the universal cover of M becomes a foliation whose space of leaves is homeomorphic with the real line R.

Theorem 3.1

If the 3-manifold M has a transversely-oriented R-covered foliation, then $\pi_1(M)$ is left-orderable.

Proof:

This can be seen by noting that $\pi_1(M)$ acts by deck transformations on the universal cover M˜ and therefore permutes the leaves of F˜. In other words, it acts on the space of leaves, assumed homeomorphic to R, and by orientation-preserving homeomorphisms because of the transverse orientation which also lifts in an equivariant way. Then apply Proposition 2.0.4.

It follows, for example, that Weeks' manifold does not support a transverselyoriented R-covered foliation. An important class of foliations are the so-called taut foliations which means that for each leaf there is a simple closed curve in the manifold intersecting that leaf and everywhere transverse to the foliation. The first examples of hyperbolic manifolds without taut foliations were given by Roberts, Shareshian and Stein [40] showing that their groups cannot act on the space of leaves, which in their case may be a possibly non-Hausdorff one-dimensional manifold. The interested reader can pursue the fascinating interplay of foliations and orderability (including circular orders) in [11] and [10].

3.2 Mappings of nonzero degree

If M and N are connected oriented manifolds of the same dimension n and $f : M \to N$ is a mapping, the degree of f is determined by the homology mapping $f_* : H_n(M;Z) \to H_n(N;Z)$. In particular, each of those top-dimensional homology -groups is canonically isomorphic to Z, coming from specified orientations. If $c \in H_n(M;Z)$ is the preferred generator, then $f_*(c) \in H_n(N;Z)$ ~= Z is the degree of f. Degree is a measure of the algebraic number of preimages of a generic point. A constant map, or more generally one with a contractible image, of course has degree zero.

It is often of interest to ask whether mappings of nonzero degree exist between given manifolds. If the target is the sphere of appropriate dimension, such maps always exist. Indeed, given any manifold M of dimension n, consider a smooth closed n-ball $B \subset M$, say a closed neighbourhood of a point. If we smash the boundary of B, as well as everything outside of B in M, to a single point, the resulting space is topologically an n-sphere. Moreover the quotient mapping $M \to S^n$ has degree one. In fact, by composing by degree d maps $S^n \to S^n$, there are maps of any given degree when the target is a sphere.

A connected sum $M_1]M_2$ always maps with degree 1 on each of its factors, simply by smashing the other factor to a point, so results assuming irreducibility often generalize. However, in general, maps of nonzero degree might not exist. Orderability gives one obstruction to their existence.

3.2.1 Theorem

Suppose M is a closed oriented irreducible 3-manifold whose fundamental group is not left-orderable and that N is a closed oriented 3-manifold whose group is left-orderable. Then maps $M \to N$ of nonzero degree do not exist.

Consider $f : M \to N$. Then our assumptions ensure that the induced map $\pi_1(M) \to \pi_1(N)$ must be trivial, because otherwise Theorem 6.0.19 would imply that $\pi_1(M)$ is left-orderable, contradicting the hypothesis. Since the induced map on fundamental groups is trivial, standard covering space theory implies f lifts to the universal cover N˜ which is noncompact. Then we have a factorization $H_3(M) \to H_3(\tilde{N}) \to H_3(N)$ of the homology map induced by f in which the middle group is trivial, because N˜ is noncompact. It follows that deg(f) = 0.

3.3. Conjectures of Waldhausen and Thurston

Group orderability is connected with certain deep conjectures about 3-manifolds due to Waldhausen and W. Thurston. A Haken 3-manifold M is one which contains an incompressible surface F, meaning a surface of genus ≥ 1 in M for which the inclusion induces an injective homomorphism $\pi_1(F) \to \pi_1(M)$. Many questions regarding 3-manifold groups had been proved for Haken manifolds, often by inductive arguments involving cutting M open along F producing a "simpler" Haken manifold. Not all 3-manifolds are Haken, but Waldhausen famously asked whether 3-manifolds are virtually Haken, meaning some finite-sheeted covering is a Haken manifold – a question which remained open for decades.

Even more audaciously, Thurston proposed a stronger conjecture for the most important, and difficult, class of 3-manifolds – hyperbolic ones. He conjectured that they are virtually fibred. A 3-manifold M is said to be fibred if there is a locally trivial fibre bundle map $M \to S^1$ with fibre a compact orientable surface. This is a very strong type of foliation in which the leaves are surfaces, all topologically equivalent, and the space of leaves is topologically a circle.

There is an exact sequence associated with fibrations, which in the case of a fibred 3-manifold M with fibre F reduces to

$1 \to \pi_1(F) \to \pi_1(M) \to \pi_1(S^1) \to 1.$

So we see that fibred 3-manifolds are Haken. Of course $\pi_1(S^1)$ is infinite cyclic and we have seen that $\pi_1(F)$ is also bi-orderable . From theorem 3.1 it follows that $\pi_1(M)$ is left-orderable if M is fibred.

Therefore there was a (faint) hope of finding a counterexample to the virtual fibred conjecture by finding a Kleinian group which is not virtually left-orderable, meaning no finite index subgroup is left-orderable. That hope was recently dashed by stunning work of Agol [1], building on results of Haglund, Wise and others, in which he proved both the virtual Haken conjecture and the virtual fibering conjecture. Moreover, he showed that if M is hyperbolic, then $\pi_1(M)$ contains a finite-index subgroup which is a right-angled Artin group, also known as RAAG. A RAAG is defined as having a finite set of generators and only relations saying that some of the generators commute with each other – a kind of blend of free group and free abelian group. Since it is known that every RAAG is bi-orderable

3.4 Triangulation

A triangulation of a topological space X is a homeomorphism from X to a simplicial complex. Let us recall that a simplicial complex K is specified by a finite set of vertices V and a finite set of simplices S Ă PpV q (the power set of V), such that if σ P S and τ Ă σ then τ P S. The combinatorial data pV,Sq is called an abstract simplicial complex. To each such data, there is an associated topological space, called the geometric realization. This is constructed inductively on d ě 0, by attaching a d-dimensional simplex ∆d for each element σ P S of cardinality d; see [Hat02]. The result is the simplicial complex K. In practice, we will not distinguish between K and the data pS,V q.

Let K “ pV,Sq be a simplicial complex. Formally, for a subset S1 Ă S, its closure is

ClpS1q “ tτ P S|τ Ď σ P S1u

The star of a simplex τ P S is

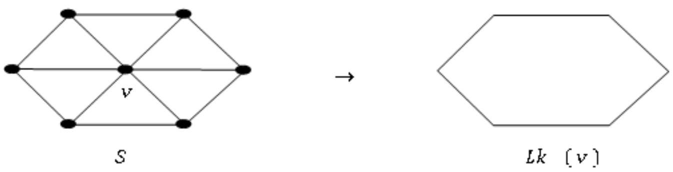
Stpτq “ tσ P S|τ Ď σu

Triangulations of manifolds

In topology, manifolds are considered in different categories, with respect to their transition functions. For example, we have, Topological manifolds if the transition functions are C0;

Smooth manifolds if the transition functions are C8;

PL (piecewise linear) manifolds if the transition functions are piecewise linear.We say that a triangulation is combinatorial if the link of every simplex (or, equivalently, of every vertex) is piecewise-linearly homomorphic to a sphere. Clearly, every space that admits a combinatorial triangulation is a manifold (in fact, a PL manifold).



4.Knot theory

Knot theory is an appealing subject because the objects studied are familiar in everyday physical space. Although the subject matter of knot theory is familiar to everyone and its problems are easily stated, arising not only in many branches of mathematics but also in such diverse fields as biology, chemistry, and physics, it is often unclear how to apply mathematical techniques even to the most basic problems. We proceed to present these mathematical techniques.

4.1 Knots

The intuitive notion of a knot is that of a knotted loop of rope. This notion leads naturally to the definition of a knot as a continuous simple closed curve in R3. Such a curve consists of a continuous function $f : [0,1] \rightarrow R^3$ with f(0) = f(1) and with f(x) = f(y) implying one of three possibilities:

1. x = y
2. x = 0 and y = 1
3. x = 1 and y = 0

Unfortunately, this definition admits pathological or so called wild knots into our studies. The remedies are either to introduce the concept of differentiability or to use polygonal curves instead of differentiable ones in the definition. The simplest definitions in knot theory are based on the latter approach.

Definition 4.1 (knot)

A knot is a simple closed polygonal curve in R^3.

The ordered set $(p_1,p_2,...,p_n)$ defines a knot; the knot being the union of the line segments $[p_1,p_2],[p_2,p_3],...,[p_{n-1},p_n]$, and $[p_n,p_1]$.

Definition 4.2 (vertices)

If the ordered set $(p_1,p_2,...,p_n)$ defines a knot and no proper ordered subset defines the same knot, the elements of the set, pi, are called the vertices of the knot.

Projections of a knot to the plane allow the representation of a knot as a knot diagram. Certain knot projections are better than others as in some projections too much information is lost.

Definition 4.3 (regular projection)

A knot projection is called a regular projection if no three points on the knot project to the same point, and no vertex projects to the same point as any other point on the knot.

Theorem 4.3.1

If a knot does not have a regular projection then there is an equivalent knot that does have a regular projection.

Figure 4: Three knot diagrams for the figure-eight knot.

A knot diagram is the regular projection of a knot to the plane with broken lines indicating where one part of the knot undercrosses the other part.

Informally, an orientation of a knot can be thought of as a direction of travel around the knot.

Definition 4.4 (oriented knot)

An oriented knot consists of a knot and an ordering of its vertices. The ordering must be chosen so that it determines the original knot. Two orderings are considered equivalent if they differ by a cyclic permutation.

The orientation of a knot on a knot diagram is represented by placing coherently directed arrows.

The connected sum of two knots, K1 and K2, is formed by removing a small arc from each knot and then connecting the four endpoints by two new arcs in such a way that no new crossings are introduced, the result being a single knot,

K = K1#K2.

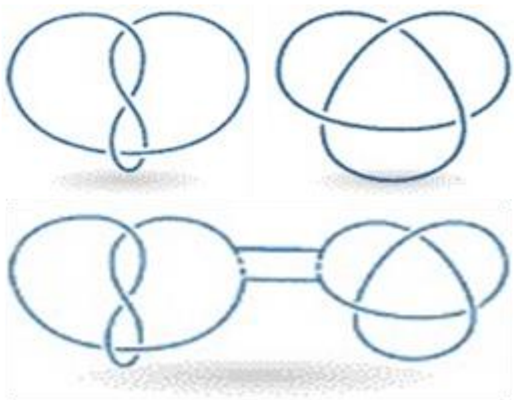


Figure 4.1 : Connected sum of the figure - eight knot and the trefoil knot.

The notion of equivalence of knots is based on their knot diagrams and the following theorem.

5. Dehn Surgery

Dehn surgery is a fundamental technique in 3-manifold topology. Indeed, we can construct any 3-manifold1 beginning with any other 3-manifold and performing Dehn surgery enough times. However, it is a highly non-trivial and widely open problem to understand what manifolds can be obtained by doing Dehn surgery once (even starting from the 'simplest' 3-manifold, namely S 3) and what knots yield a fixed manifold by surgery. Heegaard Floer theory is a relatively recent collection of powerful tools in lowdimensional topology. It has many aspects and provides invariants in many different contexts. In this paper, we are only concerned with the 3-manifold and knot invariants (defined in [18, 24, 16]). The collections of 3-manifold invariants and knot invariants are connected via the surgery formula that expresses the Heegaard Floer homology of a 3-manifold obtained by surgery on a given knot in terms of the Heegaard Floer homology data of the knot (see [22]). This makes Heegaard Floer homology an especially suitable tool for investigating questions about Dehn surgery. A natural question about Dehn surgery is whether there are manifolds that can be obtained by surgery on infinitely many distinct knots in S 3 . The answer is 'yes' – see [12] or [26]. There is still hope, however, that perhaps this does not happen for some nice classes of knots. One interesting and well-studied class of knots is that of alternating knots. At first sight, their diagrammatic definition seems to have little to do with the geometric-topological properties of these knots,

5.0.1 Theorem (Lacken by Purcell)

For any closed 3-manifold M with sufficiently large Gromov norm, there are at most finitely many prime alternating knots K and fractions p/q such that M is obtained by p/q surgery along K.

In fact, the statement about fractions p/q can be deduced, for example, from [10]. We will also show in this paper that given any manifold Y there is a universal bound on q for such fractions which also implies that they are finite in number. Using techniques that are very different from those used in [6] we are able to establish the following improvement of this theorem.

5.0.2 Theorem

Let Y be a 3-manifold. There are at most finitely many alternating knots $K \subset S^3$ such that $Y = S_{p/q}^3(K)$.

Heegaard Floer homology is also very useful in bounding genera of various surfaces. In particular, knot Floer homology determines the genus of a knot [15]. Combining this with information about surgery often allows one to put restrictions on genera of knots admitting certain surgeries. For example, if surgery on a knot K produces an L-space Y (a generalisation of lens spaces – see below for the definition), then $2g(K) - 1 \leq |H_1(Y)|$, where by g(K) we mean the genus of K

We derive a bound which is in some respects 'opposite' to the bound for Lspaces. It is a lower bound which can be non-trivial only for non-L-spaces. For the statement of the theorem below and the rest of the paper note that we work over an arbitrary field F. Heegaard Floer homology is then an F[U]-module and we denote the action of U simply by multiplication. For a rational homology sphere Y , $HF_{red}(Y)$ denotes its reduced Floer homology.

5.0.3 Theorem

For any knot $K \subset S^3$ and any $p/q \in Q$ we have Ug(K)+dg4(K)/2e · $HF_{red}({}^{S}{}_{p/q}3\ (K)) = 0$.

We remark that if K is an L-space knot, then $U^{dg4(K)/2e} \cdot HF_{red}(S_{p/q}^3(K)) = 0$. Moreover, for any N > 0 and p > 0 there is a three-manifold Y which can be obtained by a surgery on a knot in S^3 such that $U^N \cdot HF_{red}(Y)$ 6= 0 and $|H_1(Y)| = p$.

Here $g_4(K)$ is the slice genus of K. We obviously have $\lceil \frac{g_4(K)}{2} \rceil \leq \lceil \frac{g(K)}{2} \rceil$, so the theorem does give a lower bound for g(K).

A different lower bound for the knot genus producing non-L-spaces has been found by Jabuka in [4], but unlike our bound, it also depends on the denominator of the slope. Note also that there exists a manifold for which the genus of knots producing it is not bounded above [26].

More recently, Jabuka [3] has produced a new lower bound on the genus that does not involve the denominator of the slope. He also obtained the ranks of HF_d for the result of surgery on a knot in S^3. His genus bound appears to be quite different from ours.

Using the genus bound of Theorem 3 and some other considerations we are able to prove results about Seifert fibred surgery on knots in S^3. In [28] Wu (improving on the results of [19]) has proven the following (the definitions of Seifert orientation and torsion coefficients will be provided later).

5.0.4 Theorem (Wu)

Let $K \subset S^3$ be a knot. Suppose there is a rational number p/q > 0 such that $Y = S_{p/q}^3(K)$ is Seifert fibred.

If Y is a positively oriented Seifert fibred space, then all the torsion coefficients $t_i(K)$ are non-negative and HFK\(K,g(K)) is supported in even degrees. In particular, $\deg\Delta_K = g(K)$.

If Y is a negatively oriented Seifert fibred space and 0 < p/q < 3, then for all i > 0 the torsion coefficients $t_i(K)$ are non-positive. If Y is a negatively oriented Seifert fibred space, g(K) > 1 and $2g(K) - 1 > p/q$, then HFK\(K,g(K)) is supported in odd degrees. In particular, $\deg\Delta_K = g(K)$.

We are able to prove the following.

5.0.5 Theorem

Let $K \subset S^3$ be a knot. Suppose there is a rational number p/q > 0 such that $Y = S_{p/q}^3(K)$ is a negatively oriented Seifert fibred space. Then

1. Ug(K) · HFred(Y) = 0;
2. if $0 < p/q \leq 3$, then all the torsion coefficients $t_i(K)$ are non-positive (including $t_0(K)$) and $\deg\Delta_K = g(K)$;

3. more generally, if , $i \geq \lfloor \frac{\lceil p/q \rceil - \sqrt{\lceil p/q \rceil}}{2} \rfloor$ then t_i is non-positive;

- $if\ g(K) > \lfloor \frac{\lceil p/q \rceil - \sqrt{\lceil p/q \rceil}}{2} \rfloor$ 4. , then $\deg\Delta_K = g(K)$;

5. if Ub|H1(Y)|/2c · HFred(Y) = 06 then degΔK = g(K).

In all statements where $\deg\Delta_K = g(K)$ we have that HFK\(K,g(K)) is supported in odd degrees.

After the proof of Theorem 5.0.1 in Section 5, we describe negatively oriented Seifert fibred spaces Y for which the power of U needed to annihilate $HF_{red}(Y)$ gets arbitrarily large compared to the order of the first homology group.

Theorem 5.0.4 combined with the result of Wu has the following straightforward corollary.

5.1.1 Corollary

Suppose $Y = S^3_{p/q}(K)$ is a Seifert fibred rational homology sphere. If $|H_1(Y)| \leq 3$, then all the torsion coefficients of K have the same sign and $\deg\Delta_K = g(K)$.

To prove Theorems 5.0.1 and 5.0.3 we need to study the mapping cone formula, which connects the Heegaard Floer data of the knot with the Heegaard Floer homology of the manifolds obtained by surgery on it. Given a knot K in S^3 there is a doublyfiltered complex $C = CFK^\infty(K)$ associated to it. The doubly-filtered homotopy type of this complex is a knot invariant, from which all the flavours of knot Floer homology are derived.

In fact, the mapping cone formula states that given C and a certain chain homotopy equivalence which identifies $C\{i \geq 0\}$ with $C\{j \geq 0\}$ we can determine $HF^+(S^3_{p/q}(K))$ completely for any rational p/q.

In Section 3 we derive an explicit description of $HF^+(S^3_{p/q}(K))$ as an absolutely graded vector space in terms of homological data from $CFK^\infty(K)$, with no reference to the chain homotopy equivalence mentioned above. For a large part this has already been done ([11], [10], [22]), but the results are scattered across multiple papers, sometimes not in explicit form, and we consider it useful to have them collected in one place. While all the results of this section concerning positive surgeries have been shown before, as far as we are aware, the results for negative and zero surgeries (contained in subsections 3.2 and 3.3 respectively) are new.

This allows us to derive some other applications as well, a few of which we mention here.

6. Homeomorphism

In this final section I would like to touch on some known results as well as research currently under way by myself and Danny Calegari on spaces of homeomorphisms. Suppose X is a topological space with closed subset Y. We denote by Homeo(X,Y) the group of homeomorphisms X → X which are pointwise fixed on Y , the group operation being composition of functions. Homeo(X,Y) can also be endowed with a topology, which we will ignore here, but rather concentrate on algebraic and orderability properties. If X is a simplicial complex or piecewise-linear manifold and Y a PL closed subset, we consider the subgroup PL(X,Y) of homeomorphisms which are linear on each simplex of some finite subdivision of X.

Proposition

Homeo(I,∂I) is left-orderable.

This is because Homeo(I,∂I) is clearly isomorphic with $Homeo_+(R)$ which we already have seen to be left-orderable and indeed universal for countable left-orderable groups. The following was observed by Chehata [12].

Proposition (Chehata)**.** PL(I,∂I) is bi-orderable.

It should be emphasized that each element of PL(I,∂I) is a function which has only finitely many breaks where it may change slope. We can define the positive cone to be the collection of all PL homeomorphisms whose graph {(t,f(t))} in I × I has first departure from the diagonal veering above (rather than below) the diagonal.

Let us consider the 2-dimensional analogue. The following is classical.

Theorem 6.0.1(Kerekjarto, Brouwer, Eilenberg)

Homeo(I^2,∂I^2) is torsion-free. I believe it is an open question whether Homeo(I^2,∂I^2) is left-orderable.

Theorem 6.0.2 (Calegari-Rolfsen)

PL(I^2,∂I^2) is locally-indicable, and therefore left-orderable.

Here is an outline of the proof. Consider a nontrivial subgroup H of PL(I^2,∂I^2) generated by the finite set $h_1,...,h_k$ of functions. The fixed point set fix(h_i) of each generator is a finite polyhedron in I^2 containing ∂I^2, and the same may be said of the global fixed point set $fix(H) = \cap_{i=1}^k fix(h_i)$. Now we choose a point p which is on the frontier of fix(H); we can arrange that p has a neighbourhood which can be identified with $R^2 = \{(x,y)\}$ in such a way that all the functions are the identity on the x-axis and act linearly on the upper half plane. We then map H nontrivially to the "germs" of functions of H at p which according to the following lemma is a locally indicable group. It follows that H is indicable.

Lemma 6.1

The group of functions $\mathbb{R}^2_+ \to \mathbb{R}^2_+$ which are linear, and equal to the identity on the boundary, is isomorphic with the semidirect product of two locally indicable groups, and is therefore locally indicable.

Indeed such a function corresponds to a matrix with $\begin{pmatrix} 1 & s \\ 0 & r \end{pmatrix}$ $s \in R$ and $r \in R+$. Thus we have an isomorphism with the semidirect product of R as an additive group and R+ as a multiplicative group. Theorem 6.0.2 has been generalized to higher dimensions and more general manifolds in forthcoming work with Calegari.

7. Braid Group

A beautiful connection between topology and algebra is through Artin's braid groups. For each positive integer n one considers n strings in 3-space which are monotone in one direction, and disjoint, but possibly intertwined, and begin and end at specified points in two parallel planes. The product of braids is concatenation, as illustrated in Figure 1. Two braids are equivalent if one deforms to the other through a one-parameter family of braids, with endpoints fixed at all times. The identity in the n-strand braid group B_n is represented by a braid with no crossings – the strands can be taken as straight lines.

According to Artin [2] for each $n \geq 2$, B_n has generators $\sigma_1,...,\sigma_{n-1}$, in which σ_i is the simple braid in which all the strands are straight, except that the strand labelled i crosses over the strand labelled i+1. These generators are subject to the relations $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i - j| > 1$ and $\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j$ when $|i - j| = 1$.

Each n-strand braid has an associated permutation of the set {1,...,n} which records how the strands connect the endpoints of the various strands. In other words, there is a homomorphism $B_n \to S_n$, where S_n denotes the symmetric group on n elements, in which σ_i is sent to the simple permutation interchanging i and i + 1. This homomorphism is surjective – it is easy to see that any permutation of {1,...,n} can be realized by infinitely many braids (if n > 1).

Figure 7.0. The product of Δ_4,σ_1 and σ_1^{-1} in B_4

I had been working on a conjecture of J. Birman, which involved computations in the group ring ZB_n. In those calculations one always had to worry about zero divisors (for example to argue that ab = ac =⇒ b = c). Although it was wellknown that B_n is torsion-free, I didn't know if there were any zero divisors in ZB_n. My worries were over when I learned of Dehornoy's ordering of B_n and the fact that the group rings of left-orderable groups do not have zero divisors. Whether the group ring of an arbitrary torsion-free group has zero divisors is still an open question.

8. Artin's Representation

Notice that any n-braid can be formed by a finite number of **elementary** braids $\sigma_1,...,\sigma_{n-1}$, where σ_i corresponds to the geometric n-braid formed by crossing the ith string over the (i+1)th string, as depicted in figure 2.2.

i i+1

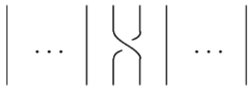
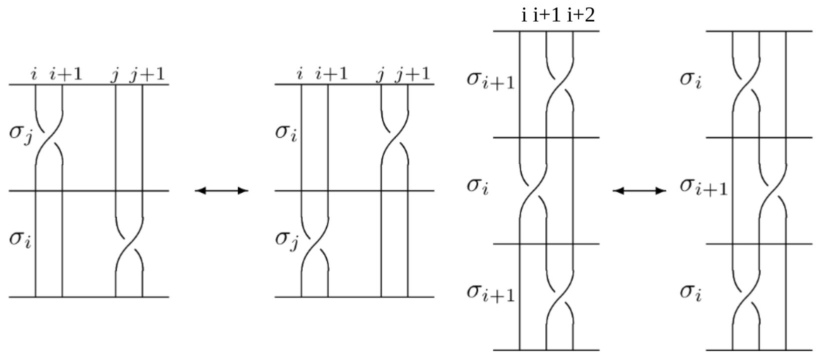


Figure 8.1: The elementary braid σ_i

We then notice that if i and j differ by more than one, then the elementary braids σ_i and σ_j commute.

Furthermore, there is an analogue for braids of the third Reidemeister move for knots and links which, written in terms of the elementary braids, becomes $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$.

i i+1 i+2



σiσj = σjσi σi+1σiσi+1 = σiσi+1σi

Figure 8.2: Relations in the elementary braids

The following theorem, due to Emil Artin, says that these two relations are sufficient to describe the n-string braid group:

9. Knot and Braids

We will present a very brief review of the theory of knots and an interesting application of the braid ordering to knot theory.

There is a close connection between the braid groups and the theory of knots and links. By a knot we mean an embedding of a circle in 3-dimensional space R^3. This models the idea of a knotted rope, but we assume the ends of the rope are attached to each other, preventing the knot from being untied by simply pulling the rope through itself. Two knots are considered equivalent if there is an isotopy (continuous family of homeomorphisms) of R^3 which at the end takes one knot to the other. One can add knots K_1 and K_2 by tying them in distant parts of the rope, thus forming the connected sum $K_1]K_2$. Figure 9.1 illustrates this.

The sum is easily seen to be commutative and associative, up to equivalence.

Thus the set of (equivalence classes of) knots forms an abelian semigroup, with

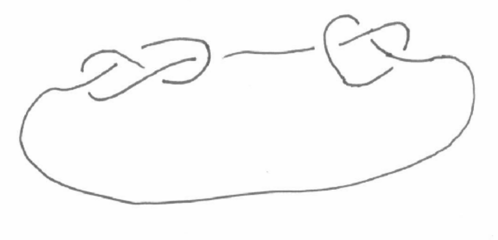


Figure 9.1. The sum 4_1] 3_1 of the figure-eight and the trefoil.

unit the trivial knot, or unknot, which is a curve equivalent to a round circle in space. There is a prime decomposition theorem: any knot K can be written $K \,\tilde{}{=}\, K_1]\cdots]K_p$ where each K_i is prime, i.e. not expressible as a sum of two nontrivial knots. Moreover, in this decomposition the terms are unique up to order. Finally, it is a theorem that there are no inverses: if $K_1]K_2$ is equivalent to the unknot, then so are both K_1 and K_2. This is one reason that braids are convenient in the study of knots, as the braids do form groups: every braid has an inverse – namely its mirror image in a plane perpendicular to the direction in which the strands are monotone.

If β is a braid, its closure $\hat{\beta}$ is a knot (or disjoint union of knots, called a link) formed by connecting the ends as indicated in Figure 9.3

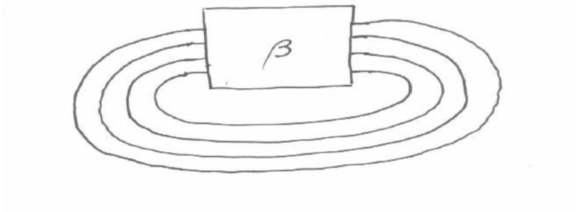


Figure 9.3. The closure of a braid

Many interesting properties of knots have arisen from this correspondence. For example the Jones polynomial [26] of a knot was discovered by considering a certain family of representations of the braid groups. There is an interesting application of the Dehornoy braid order to knot theory due to Malyutin and Netsvetaev [31].

The n-strand braid Δ_n is defined by the equation

$\Delta_n = (\sigma_1\sigma_2 \cdots \sigma_{n-1})(\sigma_1\sigma_2 \cdots \sigma_{n-2})\cdots(\sigma_1\sigma_2)(\sigma_1)$

and corresponds to the braid formed by taking n parallel strands and giving them a half-twist, as in Figure 1 for the 4-strand case. The center of B_n, for n ≥ 3 turns out to be exactly the cyclic subgroup generated by $\Delta^2{}_n$.

Theorem 9.0.1 (Malyutin and Netsvetaev)

Suppose $\beta \in B_n$ is a braid whose closure $\hat{\beta}$ is a knot. Assume that in the Dehornoy ordering of B_n one has either $\beta > \Delta_n^4 \; or \; \beta < \Delta_n^{-4}$. Then $\hat{\beta}$ is a nontrivial prime knot.

Other applications to knot theory have been found by Ito [25] which gives a lower bound on the genus of a knot (a measure of its complexity, c.f. [41]) which is the closure of a braid, in terms of the braid's place in the Dehornoy ordering. The connection between braid groups and knot theory has had profound applications, and I believe the orderability of braids will have further implications in knot theory and related areas of topology.

10. Braid groups of manifolds

To see what all this has to do with braid groups, think about the fundamental groups of the configuration spaces F_n and C_n.

The following result (due to Joan Birman[1]) suggests that the only really interesting cases of this question arise when M is a 2-manifold

10.0.1 THEOREM

Let M be a closed, smooth m-manifold. Then, for each k ∈ Z, the inclusion map

$i : F_n(M) ,\rightarrow {}^{Y}M$

n

induces a homomorphism

$$i_* : \pi_k(\mathcal{F}_n(M)) \rightarrow \prod_n \pi_k(M)$$

which is surjective if dimM > k and an isomorphism if dimM > k + 1.

This means that, unless M is a 2-manifold, the fundamental group of $F_n(M)$ is just a direct product of n copies of the fundamental group of the manifold M itself.

10.1 The braid group of the 2-manifold S^2

The braid group of the 2-manifold is similar to the braid group of the Euclidean plane, except that the points move on S^2 instead. An S^2-braid may be depicted geometrically as a braid between two concentric spheres.

The group $B_n(S^2)$ is generated by the same generators σ_i and relations as $B_n(E^2)$, but with one additional relation:

(iii) $\sigma_1\sigma_2 \ldots \sigma_n{-}_1\sigma_n{-}_1 \ldots \sigma_2\sigma_1 = 1$

This requirement says, geometrically, that the braid formed by taking the first string round behind all of the other strings and back in front of them, back to its starting position, is equivalent to the trivial braid.

By considering the geometric depiction of an S^2-braid described above, we see that this is true, since the loop may be pushed off the inner sphere without tangling with any of the other strings.

As before, we can construct a fundamental exact sequence for $B_n(S^2)$:

$$0 \longrightarrow A_n(S^2) \longrightarrow PB^i{}_n(S^2) \longrightarrow PB^j{}_{n-1}(S^2) \longrightarrow 0$$

The remark at the end of the previous subsection suggests that the braid groups of the 2-sphere and the projective plane might have some strange properties not shared by the braid groups of arbitrary 2-manifolds. This is further suggested by the following:

10.1.1 Theorem (Newwirth)

If M is either E^2 or any compact 2-manifold except P^2 or S^2 then neither $B_n(M)$ nor $PB_n(M)$ have any nontrivial elements of finite order.

So, is $B_n(S^n)$ torsion-free? Or can we find a nontrivial element of finite order?

Theorem 2.8 (Fadell/Newwirth 1962)

The word $\sigma_1\sigma_2 \ldots \sigma_{n-1}$ has order 2n in $B_n(S^2)$.

This can be seen geometrically, with a little imagination. The word $\sigma_1\sigma_2 \ldots \sigma_{n-1}$ corresponds to taking the first string over all the others to the nth position. If we do this n times, then each of the strings ends up back where it started, making a pure braid. If we then do the same thing a further n times (making 2n in total), each string winds round the remaining n − 1 strings twice. We may then utilise a move known as the 'Dirac string trick' (qv [5] for a series of diagrams depicting this operation) to untangle all n strings, resulting in a trivial braid.

In fact

$PB_2(S^2)$	= 0
$B_2(S^2)$	= Z2
$PB_3(S^2)$	= Z2
$B_3(S^2)$	is a ZS-metacyclic group of order 12

What are some of these groups $B_n(S^2)$ like? Notice that $B_n(E^2)$ is infinite for n > 1, but the previous theorem suggests that this might not necessarily be the case for the braid groups of the 2-sphere

11. Representations of braid groups

we provide a brief overview of Fox' free differential calculus, show how it may be used to construct matrix representations of automorphism groups of F_n, and then look at two examples, namely Burau and Gassner's representations of, respectively, B_n and PB_n

11.1 Free differential calculus

Let F_n be a free group of rank n, with basis $\{x_1,...,x_n\}$, and let φ be a homomorphism acting on F_n, with $\mathcal{F}_n^{\varphi}$ denoting the image of F_n under φ.

Now let $\mathbb{Z}\mathcal{F}_n^{\varphi}$ denote the integral group ring of $\mathcal{F}_n^{\varphi}$: an element of $\mathbb{Z}\mathcal{F}_n^{\varphi}$ is a sum $^{P}a_g g$, where $a_g \in Z$ and $g \in \mathcal{F}_n^{\varphi}$, with addition and multiplication defined by

$$\sum a_g g + \sum \mathcal{B}_g g = \sum (a_g + \mathcal{B}_g) g$$

$$\left(\sum a_g g\right)\left(\sum \mathcal{B}_g g\right) = \sum_g \left(\sum_h a_{gh^{-1}} \mathcal{B}_h\right) g$$

A homomorphism $\psi : F_n^{\varphi} \to F_n^{\psi\varphi}$ induces a ring homomorphism $\psi : ZF_n^{\varphi} \to ZF_n^{\psi\varphi}$. Later we will consider the cases where ψ is the abelianiser a or the trivialiser t. There is a well-defined mapping

$$\frac{\partial}{\partial x_j} : \mathbb{Z}\mathcal{F}_n \to \mathbb{Z}\mathcal{F}_n$$

given by where $g \in F_n, a_g \in Z, \varepsilon_i = \pm 1$, and $\delta_{\mu i,j}$ is the Kronecker δ.

$$\frac{\partial}{\partial x_j}\left(x_{\mu_1}^{\varepsilon_1} \ldots x_{\mu_r}^{\varepsilon_r}\right) = \sum_{i=1}^{r} \varepsilon_i \delta_{\mu_i,j} x_{\mu_1}^{\varepsilon_1} \ldots x_{\mu_i}^{\frac{1}{2}(\varepsilon_i - 1)}$$

$$\frac{\partial}{\partial x_j}\left(\sum a_g g\right) = \sum a_g \frac{\partial g}{\partial x_j}$$

The following properties follow from the definition:

(i) $\dfrac{\partial x_i}{\partial x_j} = \delta_{i,j}$

(ii) $\dfrac{\partial x_i^{-1}}{\partial x_j} = -\delta_{i,j} x_i^{-1}$

(iii) $\dfrac{\partial (wv)}{\partial x_j} = \left(\dfrac{\partial w}{\partial x_j}\right) v^{\tau} + w\left(\dfrac{\partial v}{\partial x_j}\right)$

11.2 Burau's representation of B_n

As noted before, B_n has a faithful representation as a group of automorphisms of F_n, and hence we can regard B_n as a subgroup of AutF_n.

Let Z = hti be the infinite cyclic group, and let $\psi : F_n \rightarrow Z; x_i$ 7→ t.

Then the corresponding representation, the **Burau representation** of B_n is given by:

σi 7→ kσikψ =

Ii−1	0	0	0
0	1 − t	T	0
0	1	0	0
0	0	0	In−i−1

11.3 Gassner's Representation

To represent the pure braid groups PB_n, we can simply restrict the Burau representation of B_n. But a more interesting representation exists, discovered by B.J. Gassner in 1961[4]:

Let φ be the abelianiser a. Then PB_n has a representation as a subgroup of AutF_n by the restriction of $\xi : B_n \rightarrow$ AutF_n to PB_n.

Let AF_n be the free abelian group of rank n, with basis $\{t_1,...,t_n\}$ and let a : $F_n \rightarrow AF_n$ be defined by $x_i a = t_i$.

The pure braid generators map a generator x_i of F_n into a conjugate of itself, so the requirement $x_i A_{rs} a = x_i a$ is satisfied for 1 6 i 6 n and 1 6 r < s 6 n if φ = a.

CONCLUSION

Our research has significantly advanced our comprehension of the intricate interplay between low-dimensional topology and braid groups, yielding far-reaching implications for a broad spectrum of mathematical and scientific domains. The outcomes elucidated in this study hold the promise of igniting fresh avenues of investigation and innovation in the exploration of these captivating mathematical constructs.

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