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An Inequality of Coefficients for A Subclass of Analytic Functions

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ABSTRACT:

We will construct a new type of family analytic functions with the nth derivative of coefficients of functions and its subclasses will be discussed here, by which coefficient bounds of Fekete Szego functional $|a_3 - \mu a_2^2|$ for the analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, |z| < 1$ fitting in these classes and subclasses, will be obtained.

KEYWORDS: Univalent functions, Coefficient inequality, Starlike functions, Convex functions, Close to convex functions and bounded functions.

MATHEMATICS SUBJECT CLASSIFICATION: 30C50

1. Introduction:

Let A denote the family of functions of the type

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

regular in the unit disc $E = \{z: |z| < 1|\}$. Let the family of functions of the form (1.1) which are analytic and univalent in E be denoted by S,.

Bieber Bach ([7], [8]) proved in 1916,that $|a_2| \leq 2$ for the functions $f(z)S$. Löwner [5] proved in 1923,that $|a_3| \leq 3$ for the functions $f(z)S$..

With the recognized estimates $|a_2| \leq 2$ and $|a_3| \leq 3$, naturally some relation was to be sought between a_3 and ${a_2}^2$ for the class S**,** Löwner's method was used by Fekete and Szegö [9] to prove the following well known result for the class S.

Let $f(z)\ S$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, if\ \mu \leq 0 \\ 1 + 2\, exp\, exp\left(\frac{-2\mu}{1-\mu}\right), if\ 0\ \leq \mu \\ 4\mu - 3, if \mu \geq 1 \end{cases} \tag{1.2}$$

The inequality (1.2) plays a crucial role in determining approximations of higher order coefficients for some subclasses S (See Chhichra [1], Babalola [6]).

Let us outline some subclasses of S.

We will denote by S*, the family of univalent and starlike functions

$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in A$ and satisfying the condition

$$Re\left(\frac{zg'(z)}{g(z)}\right) > 0, z \in E. \tag{1.3}$$

We denote by K, the class of univalent convex functions

$$h(z) = z + \sum_{n=2}^{\infty} c_n z^n, z\ \in A$$

and satisfying the condition

$$Re\frac{((zh'(z))'}{h'(z)} > 0, z \in E. \tag{1.4}$$

A function $f(z) \in A$ is known as close to convex function if there exists $g(z) \in S^*$ such that

$$Re\left(\frac{zf'(z)}{g(z)}\right) > 0, z \in E. \tag{1.5}$$

Kaplan [3] familiarized us with the class of close to convex functions and denoted it by C and proved that all close to convex functions are univalent.

Following are the well-known subclasses of classes of Starlike and Convex functions.

$$S^*(A,B) = \left\{ f(z) \in A; \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, -1 \le B < A \le 1, z \in E \right\} \tag{1.6}$$

$$K(A,B) = \left\{ f(z) \in A; \frac{\left(zf'(z)\right)'}{f'(z)} \prec \frac{1+Az}{1+Bz}, -1 \le B < A \le 1, z \in E \right\} \tag{1.7}$$

It is obvious that $S^*(A,B)$ is a subclass of S^* and $K\ (A,B)$ is a subclass of K.

We introduced a new class

$$P^* = \left\{ f(z) \in A; \frac{z(f'(z) + z(f'(z))')}{f(Z) + zf'(z)} \prec \frac{1+z}{1-z}; z \in E \right\}$$

Now, we introduce a new subclass

$P^*(A,B)$=$\left\{ f(z) \in A; \frac{z(f'(z)+z(f'(z))')}{f(Z)+zf'(z)} \prec \left[\frac{1+Aw(z)}{1+Bw(z)}\right]; z \in E \right\}$

Symbol $\prec$ stands for subordination, which we describe as follows:

Principle of Subordination: Let $f(z)$ and $F(z)$ be two functions analytic in E. Then $f(z)$ is called subordinate to F(z) in E if there exists a function $w(z)$ analytic in E satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = F(w(z))$; $z\ E$ and we write $f(z)\ \prec\ F(z)$.

By U, we denote the class of analytic bounded functions of the form

$$w(z) = \sum_{n=1}^{\infty} d_n z^n, w(0) = 0, |w(z)| < 1. \tag{1.8}$$

It is known that

$$|d_1| \le 1, |d_2| \le\ 1 - |d_1|^2. \tag{1.9}$$

2. PRELIMINARY LEMMAS:

For $0\ <\ c\ <\ 1$, we write

$$w(z)\ = \left(\frac{c+z}{1+cz}\right)$$

so that

$$\frac{1+Aw(z)}{1+Bw(z)}\ =\ 1\ +\ (A-B)c_1 z +\ (A-B)(c_2\ - Bc_1^2)z^2 + - - - \tag{2.1}$$

3. MAIN RESULTS

THEOREM 3.1: Let $f(z) \in P^*$, then

$$|a_3 - \mu {a_2}^2| \le \begin{cases} \frac{(A-2B)(A-B)}{4} - \frac{4(A-B)^2}{9}\mu\ ;\ if\ \ \mu \le \frac{9[(A-2B)-1]}{16(A-B)} \\ \frac{A-B}{4}\ ;\ \ if\ \frac{9[(A-2B)-1]}{16(A-B)} \le \mu \le\ \frac{9[(A+2B)+1]}{16(A-B)} \\ \frac{4(A-B)^2}{9}\mu\ \ - \frac{(A-2B)(A-B)}{4};\ \text{if}\, \mu\ \ge \frac{9[(A+2B)+1]}{16(A-B)} \end{cases}$$

The results are sharp.

Proof: By definition of $f(z) \in S_n^*\,(A;B)$, we have

$$\frac{z(f'(z) + z(f'(z))')}{f(Z) + zf'(z)} = \left[\frac{1+Aw(z)}{1+Bw(z)}\right]; w(z) \in U. \tag{3.4}$$

Expanding the series (3.4), we get

$$\{2 + 6a_2 z + 12a_3 z^2 + \cdots\} = \{2 + [3a_2 + 2(A-B)c_1]z + [4a_3 + 3a_2\ (A-B)c_1 + 2(A-B)(c_{\ 2} - B{c_1}^2)z^2\ \ldots\} \tag{3.5}$$

Identifying terms in (3.5), we get

$$a_2 = \frac{2(A-B)}{3}c_1 \tag{3.6}$$

$$a_3 = \frac{(A-B)}{4}c_2 + \frac{(A-B)(A-2B)}{4}c_1^2 \tag{3.7}$$

From (3.6) and (3.7), we obtain

$$a_3 - \mu a_2^2 = \frac{(A-B)}{4}c_2 + \frac{4(A-B)^2}{9}\left\{\frac{9(A-2B)}{16(A-B)} - \mu\right\}c_1^2 \tag{3.8}$$

Taking absolute value, (3.8) can be rewritten as

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)}{4}|c_2| + \frac{4(A-B)^2}{9}\left|\frac{9(A-2B)}{16(A-B)} - \mu\right| |c_1{}^2| \tag{3.9}$$

Using (1.9) in (3.9), we get

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)}{4}(1-|c_1|^2) + \frac{4(A-B)^2}{9}\left|\frac{9(A-2B)}{16(A-B)} - \mu\right| |c_1|^2$$
$$= \frac{A-B}{4} + \frac{4(A-B)^2}{9}\left[\left|\frac{9(A-2B)}{16(A-B)} - \mu\right| - \frac{9}{16(A-B)}\right]|c_1|^2 \tag{3.10}$$

Case I: $\mu \leq \frac{9(A-2B)}{16(A-B)}$

(3.10) can be rewritten as

$$|a_3 - \mu a_2^2| \leq \frac{A-B}{4} + \frac{4(A-B)^2}{9}\left\{\frac{9[(A-2B)-1]}{16(A-B)} - \mu\right\}|c_1|^2 \tag{3.11}$$

Subcase I (a): $\mu \leq$ -$\frac{9[(A-2B)-1]}{16(A-B)}$

Using (1.9), (3.11) becomes

$$|a_3 - \mu a_2^2| \leq \frac{(A-2B)(A-B)}{4} - \frac{4(A-B)^2}{9}\mu \tag{3.12}$$

Subcase I (b): $\mu \geq \frac{9[(A-2B)-1]}{16(A-B)}$

We obtain from (3.11)

$$|a_3 - \mu a_2^2| \leq \frac{A-B}{4} \tag{3.13}$$

Case II: $\mu \geq \frac{9(A-2B)}{16(A-B)}$

Preceding as in case I, we get

$$|a_3 - \mu a_2^2| \leq \frac{A-B}{4} + \frac{4(A-B)^2}{9}\left[\mu - \frac{9[(A-2B)+1]}{16(A-B)}\right]|c_1|^2 \tag{3.14}$$

Subcase II (a): $\mu \leq \frac{9[(A-2B)+1]}{16(A-B)}$

(3.14) takes the form

$$|a_3 - \mu a_2^2| \leq \frac{A-B}{4} \tag{3.15}$$

Combining the results of subcases I(b) and II(a), we can write

$$|a_3 - \mu a_2^2| \leq \frac{A-B}{4}\ ;\ if\ \frac{9[(A-2B)-1]}{16(A-B)} \leq \mu \leq \frac{9[(A-2B)+1]}{16(A-B)} \tag{3.16}$$

Subcase II (b): $\mu \geq \frac{9[(A-2B)+1]}{16(A-B)}$

Preceding as in subcase I (a), we get

$$|a_3 - \mu a_2^2| \leq \frac{4(A-B)^2}{9}\mu - \frac{(A-2B)(A-B)}{4} \tag{3.17}$$

Combining (3.12), (3.16) and (3.17), the theorem is established.

Corollary 3.2: Putting $A = 1, B = -1$ and $n = 0$ in the theorem, we get

$$|a_3 - \mu a_2{}^2| \leq \begin{cases} \frac{3}{2} - \frac{16}{9}\mu\ ;\ if\ \mu \leq \frac{9}{16} \\ \frac{1}{2}\ ;\ if\ \frac{9}{16} \leq \mu \leq \frac{9}{8} \\ \frac{16}{9}\mu - \frac{3}{2};\ \text{if}\ \mu \geq \frac{9}{8} \end{cases}$$

These approximations were derived by Keogh and Merkes [6] and are outcomes for the class of univalent starlike functions.

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