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On Finding Integer Solutions to System of Four Diophantine Equations

$$a+2b=\alpha^2,\ a+2c=\beta^2,\ b+c=\gamma^2,\ a+b+c=2\delta^3$$

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Abstract

This paper aims at determining explicitly three non-zero distinct integers a,b,c such that $a+2b=\alpha^2,a+2c=\beta^2,b+c=\gamma^2,a+b+c=2\delta^3$. Different methods have been considered to obtain the three required integer a,b,c.

Keywords: System of four Diophantine equations, Ternary cubic equation, Diophantine problem, Integer solutions

Introduction

Systems of indeterminate quadratic equations of the form ax + c = u^2, bx + d = v^2 where a, b, c, d are non-zero distinct constants, have been investigated for solutions by several authors [1, 2] and with a few possible exceptions, most of the them were primarily concerned with rational solutions. Even those existing works wherein integral solutions have been attempted, deal essentially with specific cases only and do not exhibit methods of finding integral solutions is a general form. In [3], a general form of the integral solutions to the system of equations ax + c = u^2 , bx + d = v^2 where a, b, c, d are non-zero distinct constants is presented when the product ab is a square free integer whereas the product cd may or may not a square integer. For other forms of system of double, triple Diophantine equations, one may refer [4-20].

This communication concerns with yet another interesting system of four Diophantine equations

$a+2b=\alpha^2,a+2c=\beta^2,b+c=\gamma^2,a+b+c=2\delta^3$ for its infinitely many non-zero distinct integer solutions.

Method of Analysis

Illustration 1:

The Diophantine problem under consideration is to obtain three non-zero distinct integers a,b,c such that

$$a+2b=\alpha^2 \quad (1)$$

$$a+2c=\beta^2 \quad (2)$$

$$b+c=\gamma^2 \quad (3)$$

$$a+b+c=2\delta^3 \quad (4)$$

Eliminating b and c between (1) to (3),one has

$$2a=\alpha^2+\beta^2-2\gamma^2 \quad (5)$$

From (3) and (4), we get

$$a=2\delta^3-\gamma^2 \quad (6)$$

Substituting of (6) in (5) leads to

$$\alpha^2 + \beta^2 = 4\delta^3 \tag{7}$$

which is satisfied by

$$\alpha = 2p\left(p^2 + q^2\right) \tag{8}$$

$$\beta = 2q\left(p^2 + q^2\right) \tag{9}$$

$$\delta = p^2 + q^2 \tag{10}$$

Substituting (6), (8) and (10) in (1), it is seen that

$$2b = 2\left(p^2 + q^2\right)^2\left(p^2 - q^2\right) + \gamma^2$$

Since our interest is on finding integer solutions,

Taking $\gamma = 2r$ (11)

in the above equation, we get

$$b = \left(p^2 + q^2\right)^2\left(p^2 - q^2\right) + 2r^2 \tag{12}$$

Similarly, substituting (6), (9), (10) in (2) and using (11), we get

$$c = \left(p^2 + q^2\right)^2\left(q^2 - p^2\right) + 2r^2 \tag{13}$$

Using (10), (11) in (6), we get

$$a = 2\left(p^2 + q^2\right)^3 - 4r^2 \tag{14}$$

Thus, (12), (13), (14) satisfy the system of four equations (1) to (4).

A few numerical examples are presented in the table below

Table 1

p	Q	r	a	b	c	a+2b	a+2c	b+c	a+b=c
1	2	1	246	-73	77	100	400	4	250
3	4	2	31234	-4367	4383	22500	40000	16	31250
5	6	3	453926	-40913	40949	372100	535824	36	453962
7	8	4	2885730	-191503	191567	2502724	3268864	64	2885794

Note: 1

It is worth to note that (7) is, in general, satisfied by

$$\alpha = 2^{3k-2}p\left(p^2 + q^2\right)$$

$$\beta = 2^{3k-2}q\left(p^2 + q^2\right)$$

$$\delta = 2^{2k-2}\left(p^2 + q^2\right)$$

In this case, the corresponding integer solutions to the system of 4 equations (1) to (4) are given by

$$a = 2.2^{6k-6}\left(p^2 + q^2\right)^3 - \gamma^2$$

$$b = 2^{6k-5}p^2\left(p^2 + q^2\right)^2 - 2^{6k-6}\left(p^2 + q^2\right)^3 + 2r^2$$

$$c = 2^{6k-4}q^2\left(p^2 + q^2\right)^2 - 2^{6k-5}\left(p^2 + q^2\right)^3 + 4r^2$$

Illustration 2:

Assume $\delta = A^2 + B^2$ (15)

Write the integer 4 as the R.H.S of (7) as

$$4 = (2i)(-2i) \tag{16}$$

Substituting (15) and (16) in (7) and employing the method of factorization

$$\alpha + i\beta = 2i\left[\left(A^3 - 3AB^2\right) + i\left(3A^2B - B^3\right)\right]$$

from which we get

$$\alpha = 2B^3 - 6A^2B \tag{17}$$

$$\beta = 2A^3 - 6AB^2 \tag{18}$$

Substituting (15),(17),(18)in (1),(2),(6) and taking $\gamma = 2r$, the corresponding values of a,b,c satisfying the system of four equations (1) to (4) are as below:

$$a = 2\left(A^2 + B^2\right)^3 - 4r^2$$

$$b = 2r^2 + B^6 - A^6 + 15A^4B^2 - 15A^2B^4$$

$$c = 2r^2 + A^6 - B^6 + 15A^4B^2 - 15A^2B^4$$

Note: 2

The integer 4 on the R.H.S of (7) is also represented as

1. $4 = \dfrac{(6+8i)(6-8i)}{25}$
2. $4 = \dfrac{(8+6i)(8-6i)}{25}$

References:

1. L.E. Dickson, History of the theory of Numbers, Vol.II, Chelsea publishing company, New York, 1952.
2. B. Batta and A.N. Singh, History of Hindu Mathematics, Asia Publishing House, 1938.
3. M.A. Gopalan, S. Vidhyalakshmi, N. Thiruniraiselvi, " On Interesting Integer Pairs", IJMRME, Vol-1, Iss-1, 2015, Pp 320-323.
4. M.A. Gopalan, S. Vidhyalakshmi, J. Shanthi, " On Interesting Diophantine Problem", IJMRME, Vol-1, Iss-1, 2015, Pp 168-170.
5. M.A. Gopalan, S. Vidhyalakshmi, J. Shanthi, " On Interesting Diophantine Problem", Scholar Bulletin, Vol-1, Iss-7, 2015, Pp 166-168.
6. M.A. Gopalan, S. Vidhyalakshmi, J. Shanthi, A. Rukmani" On Interesting Diophantine Problem on Triples-II", IJCMS, Vol-2, Iss-2, February 2016, Pp 130-133.
7. S. Vidhyalakshmi, T. Mahalakshmi, H. Ayesha Begum, A. Prathiba, M.A. Gopalan," On Two Interesting Systems of Triple Diophantine Equations With Five Unknowns", IJRAR, Vol-6, Iss-2, June 2019, Pp 637-645.
8. M.A. Gopalan, S. Vidhyalakshmi, E.Premalatha., Three fascinating paits, Journal of Asian Scientific Research., Vol 7, No 6, 2017, Pp 231-239.
9. S. Vidhyalakshmi, T. Mahalakshmi, J. Shanthi, M.A. Gopalan," On Two Interesting Systems of Diophantine Equations", Journal of Interdisciplinary Cycle Research, Vol-XI, Iss-XI, November 2019, Pp 692-695.
10. N.Thiruniraiselvi, C.Dhanapriya, M.A.Gopalan, On Simultaneous equations ., Science, Technology and Development., Vol IX issue III, March 2020., Pp 132-143.
11. S.Vidyalakshmi , M. A. Gopalan, E.Premlatha , on two remarkable triplets, Fundamental Journal of Mathematics and Mathematical Sciences ., Vol5, issue 2, 2016, Pp 91-95.
12. M. A. Gopalan , Sharadha Kumar, On the system of Double Equations, IJMR, Vol 5(9), 2019, Pp 91-95.
13. Sharadha Kumar , M. A. Gopalan, On the simultaneous equations,IJMR, Vol 4(2), 2019, Pp 1-3.
14. A. Vijayashankar , Sharda Kumar, M. A. Gopalan, On the pair of equations, Journal of Mathematics and System Science, Vol 9, 2019, Pp 115-118.
15. S. Vidyalakshmi , E. Premlatha, Presenna, V. Krithika, Construction of a special integer triplet-I, IJPMS, Vol 16, 2016, Pp 24-29.
16. A.Vijayashankar, M.A. Gopalan , V. kiruthika, On diophantine problem, IJMRD, Vol 3(9), 2016, Pp137-139.
17. M. A. Gopalan, S.Vidyalakshmi , E.Premlatha, R.Sridevi ,An interesting diophantine problem on triples II, Journal of Computing Technologies, Vol 4(11), Pp 18-20.

18. M. A. Gopalan, S.Vidyalakshmi , E.Premlatha , On interesting diophantine problem, Jamal Academic Research journal , special issue, 2016, Pp287-290.
19. A. Vijayashankar , Sharda Kumar, M. A. Gopalan, On the simultaneous equation , IJEI, Vol 8(4), 2019, Pp65-68.
20. S.Vidyalakshmi , M. A. Gopalan, Aarthy Thangam, Three special system of double type and equations , International Journal of recent scientific research, Vol 8(12),2017, Pp 22292-22296.