WWW.IJRPR.COM

**International Journal of Research Publication and Reviews**

Journal homepage: www.ijrpr.com ISSN 2582-7421

# Order and Convergence of Order 7th Numerical Scheme for the Solution of First Order System of IVP of Ordinary Differential Equations

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**Abstract**

This work proposed order and convergence of order seventh numerical scheme for integrating first order stiff initial value problem. All the necessary and sufficient condition for convergence of multistep method are satisfied; the order of the method was analysed, the stability properties are investigated, the scheme is found to be zero stable and consistent. Hence, the scheme is convergent.

**Keyword**: Stiff, Order, Convergence, Ordinary Differential equation, IVPs

## Introduction

Block backward differentiation formula is one of the reliable block numerical methods for obtaining solutions of stiff initial value problems. Backward differentiation formula was first discovered by Curtiss & Hirschfelder (1952), in his method integration of stiff equations, Cash (1980) extended the work of Curtiss, with integration of stiff system of ODEs using extended backward differentiation formula, Milner (1953) discovered block numerical solution of differential equation, Brugano (1998) with solving differential problem by multistep method, Chu and Hamilton (1987) with parallel solution of ODE's by multistep method, Dalquish (1974) with problem related to numerical method, an order five implicit 3-step block method for solving ordinary differential equation (Yahaya *et al*., 2013), Implicit r-point block backward differentiation formula for solving first- order stiff ODEs (Ibrahim *et al*.,2007), a new variable step size block backward differentiation formula for solving stiff initial value problems (Suleiman *et al*., 2013), a new fifth order implicit block method for solving first order stiff ordinary differential equations by (Musa *et al* 2014), Musa et al (2016); Diagonally implicit super class of block backward differentiation formula for solving Stiff IVPs, Sagir *et al* (2012, 2013, 2022); an accurate computation of block hybrid method for solving stiff ODEs, One-leg Multistep Method for first Order Differential Equations (Fatunla, 1984), Abdullahi et al (2021, 2021, 2022, 2022); Enhanced 3 point fully implicit super class of block backward differentiation formula for solving first order stiff initial value problems, Order and Convergence of the Enhanced 3-Point Fully Implicit Super Class of Block Backward Differentiation Formula for Solving First Order Stiff Initial Value Problems among other researches. All the method highlighted above possesses different degree of accuracy in one way or the other. However, some numerical method have good accuracy but, no advantage whatsoever when it comes to the issue of computational time. Initial value problem found in engineering and sciences need a scheme that is not only convergent. But, converge faster within a minimum number of iteration. This research aimed at testing the required criteria for the convergence of a numerical scheme for the solution of the system of IVPs of ODEs.

## Material and Methods

Consider the block backward differentiation formula of the form

$$\sum_{j=0}^{7} \alpha_j \, y_{n+j-3} = h\beta_{k} f_{n+k-3} \qquad k = 1,2,3,4 \tag{1}$$

The implicit four point method (1) is constructed using a linear operator $L_i$. To derive the four point, define the linear operator $L_i$ associated with (1) as

$$L_i[y(x_n),h)]: \alpha_0\, y_{n-3} + \alpha_1\, y_{n-2+}\alpha_2\, y_{n-1} + \alpha_3\, y_n + \alpha_4\, y_{n+1} + \alpha_5\, y_{n+2} + \alpha_6\, y_{n+3} + \alpha_7\, y_{n+4} - h\beta_k f_{n+k-3} = 0 \qquad k = i = 1,2,3,4 \tag{2}$$

To derive the first, second, third, and fourth points as $y_{n+1},\ y_{n+2},\ y_{n+3}\ and\ \ y_{n+4}$ respectively Using Taylor series expansion in (2) and normalizing $\alpha_3 = 1$, $\alpha_4 = 1$, $\alpha_5 = 1$and $\alpha_6 = 1$ as coefficient's of the four points, $k = 1$, $k = 2$, $k = 3$ and $k = 4$respectively. To obtain

$$\begin{aligned}
y_{n+1} &= -\frac{1298881}{341643939} y_{n-3} + \frac{341643939}{569406565} y_{n-2} - \frac{72003623}{113881313} y_{n-1} + \frac{426060731}{341643939} y_n + \frac{6274637}{16268759} y_{n+2} - \frac{143998979}{1708219695} y_{n+3} + \frac{1847955}{113881313} y_{n+4} - \frac{9603792}{113881313} f_{n-2} \\
y_{n+2} &= -\frac{79696}{845265} y_{n-3} + \frac{41929759}{9861425} y_{n-2} - \frac{68414023}{3944570} y_{n-1} + \frac{189894686}{5916855} y_n - \frac{7210474}{394457} y_{n+1} + \frac{21582821}{59168550} y_{n+3} + \frac{14016}{1972285} y_{n+4} + \frac{19789614}{1972285} f_{n-1} \\
y_{n+3} &= \frac{70450}{1797393} y_{n-3} - \frac{1295843}{1198262} y_{n-2} + \frac{5593225}{599131} y_{n-1} + \frac{676840}{105729} y_n - \frac{11495780}{599131} y_{n+1} + \frac{6496015}{1198262} y_{n+2} + \frac{42690}{599131} y_{n+4} + \frac{845710}{46087} f_n \\
y_{n+4} &= -\frac{338687}{348237} y_{n-3} - \frac{353855969}{77076456} y_{n-2} + \frac{2326014617}{19269114} y_{n-1} - \frac{4938738481}{115614684} y_n + \frac{1117145237}{19269114} y_{n+1} - \frac{11296250177}{77076456} y_{n+2} + \frac{495749336}{28903671} y_{n+3} + \frac{951570371}{3211519} f_{n+1}
\end{aligned} \tag{3}$$

## Order of the Method

In this section, we derive the order of the methods (3). The method can be written as

$$\sum_{j=0}^{1} C_j^* Y_{m+j-1} = h\sum_{j=0}^{1} D_j^* Y_{m+j-1}, \tag{4}$$

where $C_0^*, C_1^*, D_0^*$ and $D_1^*$ are block matrices defined by

$C_0^* = [C_0, C_1, C_2, C_3]$, $C_1^* = [C_4, C_5, C_6, C_7]$, $D_0^* = [D_0, D_1, D_2, D_3]$, $D_1^* = [D_4, D_5, D_6, , D_7]$ $C_0^*$, $C_1^*$, $D_0^*$ and $D_1^*$ are square matrices and $Y_{m-1}, Y_m, F_{m-1}$ and $F_m$ are column vectors defined as

$$Y_m = \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \end{bmatrix} = \begin{bmatrix} y_{3m+1} \\ y_{3m+2} \\ y_{3m+3} \\ y_{3m+4} \end{bmatrix},\ Y_{m-1} = \begin{bmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} = \begin{bmatrix} y_{3(m-1)+1} \\ y_{3(m-1)+2} \\ y_{3(m-1)+3} \\ y_{3(m-1)+4} \end{bmatrix},\ F_{m-1} = \begin{bmatrix} f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix} = \begin{bmatrix} f_{3(m-1)+1} \\ f_{3(m-1)+2} \\ f_{3(m-1)+3} \\ f_{3(m-1)+4} \end{bmatrix}$$

$$F_m = \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{bmatrix} = \begin{bmatrix} f_{3m+1} \\ f_{3m+2} \\ f_{3m+3} \\ f_{3m+4} \end{bmatrix} \tag{5}$$

Thus, (3) can be rewritten as

$$\begin{bmatrix} -\frac{1298881}{341643939} & \frac{341643939}{569406565} & -\frac{72003623}{113881313} & \frac{426060731}{341643939} \\ -\frac{79696}{845265} & \frac{41929759}{9861425} & -\frac{68414023}{3944570} & \frac{189894686}{5916855} \\ \frac{70450}{1797393} & -\frac{1295843}{1198262} & \frac{5593225}{599131} & \frac{676840}{105729} \\ -\frac{338687}{348237} & -\frac{353855969}{77076456} & \frac{2326014617}{19269114} & -\frac{49388481}{115614684} \end{bmatrix} \begin{bmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} +$$

$$\begin{bmatrix} 1 & -\frac{6274637}{16268759} & \frac{143998979}{1708219695} & -\frac{9603792}{113881313} \\ \frac{7210474}{394457} & 1 & -\frac{21582821}{59168550} & -\frac{14016}{1972285} \\ \frac{11495780}{599131} & -\frac{6496015}{1198262} & 1 & -\frac{42690}{599131} \\ -\frac{1117145237}{19269114} & \frac{11296250177}{77076456} & -\frac{495749336}{28903671} & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \end{bmatrix} = h \begin{bmatrix} 0 & \frac{9603792}{113881313} & 0 & 0 \\ 0 & 0 & \frac{19789614}{1972285} & 0 \\ 0 & 0 & 0 & \frac{845710}{46087} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix} + h$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{951570371}{3211519} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{bmatrix} \tag{6}$$

From the (6) we have

$$C_0^* = \begin{bmatrix} -\frac{1298881}{341643939} & \frac{341643939}{569406565} & -\frac{72003623}{113881313} & \frac{426060731}{341643939} \\ -\frac{79696}{845265} & \frac{41929759}{9861425} & -\frac{68414023}{3944570} & \frac{189894686}{5916855} \\ \frac{70450}{1797393} & -\frac{1295843}{1198262} & \frac{5593225}{599131} & \frac{676840}{105729} \\ -\frac{338687}{348237} & -\frac{353855969}{77076456} & \frac{2326014617}{19269114} & -\frac{49388481}{115614684} \end{bmatrix} \tag{7}$$

$$C_1^* = \begin{bmatrix} 1 & -\frac{6274637}{16268759} & \frac{143998979}{1708219695} & -\frac{9603792}{113881313} \\ \frac{7210474}{394457} & 1 & -\frac{21582821}{59168550} & -\frac{14016}{1972285} \\ \frac{11495780}{599131} & -\frac{6496015}{1198262} & 1 & -\frac{42690}{599131} \\ -\frac{1117145237}{19269114} & \frac{11296250177}{77076456} & -\frac{495749336}{28903671} & 1 \end{bmatrix} \tag{8}$$

$$D_0^* = \begin{bmatrix} 0 & \frac{9603792}{113881313} & 0 & 0 \\ 0 & 0 & \frac{19789614}{1972285} & 0 \\ 0 & 0 & 0 & \frac{845710}{46087} \\ 0 & 0 & 0 & 0 \end{bmatrix} D_1^* = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{951570371}{3211519} & 0 & 0 & 0 \end{bmatrix} \tag{9}$$

Where

$$C_0 = \begin{bmatrix} -\frac{1298881}{341643939} \\ -\frac{79696}{845265} \\ \frac{70450}{1797393} \\ -\frac{338687}{348237} \end{bmatrix} C_1 = \begin{bmatrix} \frac{341643939}{569406565} \\ -\frac{41929759}{9861425} \\ -\frac{1295843}{1198262} \\ -\frac{353855969}{348237} \end{bmatrix} C_2 = \begin{bmatrix} -\frac{72003623}{113881313} \\ -\frac{68414023}{3944570} \\ \frac{5593225}{599131} \\ \frac{2326014617}{19269114} \end{bmatrix} C_3 = \begin{bmatrix} \frac{426060731}{341643939} \\ \frac{1898894686}{5916855} \\ \frac{676840}{105729} \\ -\frac{49388481}{115614684} \end{bmatrix} \tag{10}$$

$$C_4 = \begin{bmatrix} 1 \\ \frac{7210474}{394457} \\ \frac{11495780}{599131} \\ -\frac{1117145237}{19269114} \end{bmatrix} C_5 = \begin{bmatrix} -\frac{6274637}{16268759} \\ 1 \\ -\frac{6496015}{1198262} \\ \frac{11296250177}{77076456} \end{bmatrix} C_6 = \begin{bmatrix} \frac{143998979}{1708219695} \\ -\frac{21582821}{59168550} \\ 1 \\ -\frac{495749336}{28903671} \end{bmatrix} C_7 = \begin{bmatrix} -\frac{9603792}{113881313} \\ -\frac{14016}{1972285} \\ -\frac{42690}{599131} \\ 1 \end{bmatrix} \tag{11}$$

$$D_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} D_1 = \begin{bmatrix} \frac{9603792}{113881313} \\ 0 \\ 0 \\ 0 \end{bmatrix} D_2 = \begin{bmatrix} 0 \\ \frac{19789614}{1972285} \\ 0 \\ 0 \end{bmatrix} D_3 = \begin{bmatrix} 0 \\ 0 \\ \frac{845710}{46087} \\ 0 \end{bmatrix} D_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{951570371}{3211519} \end{bmatrix} \tag{12}$$

$$D_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{951570371}{3211519} \end{bmatrix} D_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} D_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} D_7 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{13}$$

**Definition:** (Order of the method) the order of the block method (3) and its associated linear operator are given by

$$L[\mathrm{y}(\mathrm{x});\mathrm{h}] = \sum_{\mathrm{j}=0}^{7}[\mathrm{C_j}\mathrm{y}(\mathrm{x}+\mathrm{jh})] - \mathrm{h}\sum_{\mathrm{j}=0}^{7}[\mathrm{D_j}y'(\mathrm{x}+\mathrm{jh})] \tag{14}$$

where p is unique integer such that

$\mathrm{E_q} = 0,\ \mathrm{q} = 0,1,\ldots\ \mathrm{p}$ and $\mathrm{E_{p+1}} \neq 0$, where the $\mathrm{E_q}$ are constant matrix

With

$$E_0 = \sum_{J=0}^{7} C_J = 0 \tag{16}$$
$$E_1 = \sum_{J=0}^{7}[jC_j - 2D_j] = 0 \tag{17}$$
$$E_2 = \sum_{J=0}^{7}\left[\frac{1}{2!}j^2C_j - 2jD_j\right] = 0 \tag{18}$$
$$E_3 = \sum_{J=0}^{7}\left[\frac{1}{3!}j^3C_j - 2\frac{1}{2!}j^2D_j\right] = 0 \tag{19}$$
$$E_4 = \sum_{J=0}^{7}\left[\frac{1}{4!}j^4C_j - 2\frac{1}{3!}j^3D_j\right] = 0 \tag{20}$$
$$E_5 = \sum_{J=0}^{7}\left[\frac{1}{5!}j^5C_j - 2\frac{1}{4!}j^4D_j\right] = 0 \tag{21}$$
$$E_6 = \sum_{J=0}^{7}\left[\frac{1}{6!}j^6C_j - 2\frac{1}{5!}j^5D_j\right] = 0 \tag{22}$$
$$E_7 = \sum_{J=0}^{7}\left[\frac{1}{6!}j^6C_j - 2\frac{1}{5!}j^5D_j\right] = 0 \tag{23}$$
$$E_8 = \sum_{J=0}^{7}\left[\frac{1}{7!}j^7C_j - 2\frac{1}{6!}j^6D_j\right] \neq 0 \tag{23}$$

Therefore, the method (3) is of order 7, with error constant as: $E_8 = \begin{bmatrix} -\frac{210}{8293585} \\ \frac{324}{3184255} \\ -\frac{981}{6926402} \\ \frac{563}{5947583} \end{bmatrix}$

## Convergence of the Method

In this section, we apply the theorem on convergence by Henrici (1962) to analysed the convergence of the method (3)

**Theorem (1)**: Henrici (1962) stated the following conditions for convergence of Linear Multi-Step Method (LMM):

1. A necessary condition for convergence of the Linear Multi-step Method (3) is that the modulus of none of the root of the associated polynomial $\rho(\xi)$ exceeds one, and that the roots of modulus one is simple. The condition, thus imposed on $\rho(\xi)$ is called the condition of zero stability.

2. A necessary condition for convergence of the Linear Multi-step Method (3) is that the order of the associated difference operator be at least one. The condition that the order $\rho \geq 1$, is called the condition of consistency.

To investigate the convergence of (3), the method need to meet conditions (1) and (2) in the stated theorem.

### *Zero Stability of the Method*

**Definition:** (Zero-Stable) A linear multistep method is said to be zero stable if no root of its first characteristics polynomial has modulus greater than one and that any root with modulus one, is simple.

The method (3) is converted into matrix form as:

$$\begin{bmatrix} 1 & -\frac{6274637}{16268759} & \frac{143998979}{1708219695} & -\frac{9603792}{113881313} \\ \frac{7210474}{394457} & 1 & -\frac{21582821}{59168550} & -\frac{14016}{1972285} \\ \frac{11495780}{599131} & -\frac{6496015}{1198262} & 1 & -\frac{42690}{599131} \\ -\frac{1117145237}{19269114} & \frac{11296250177}{77076456} & -\frac{495749336}{28903671} & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \end{bmatrix} = h$$
$$\begin{bmatrix} \frac{1298881}{341643939} & -\frac{341643939}{569406565} & \frac{72003623}{113881313} & -\frac{426060731}{341643939} \\ \frac{79696}{845265} & -\frac{41929759}{9861425} & \frac{68414023}{3944570} & -\frac{189894686}{5916855} \\ -\frac{70450}{1797393} & \frac{1295843}{1198262} & -\frac{5593225}{599131} & -\frac{676840}{105729} \\ \frac{338687}{348237} & \frac{353855969}{77076456} & -\frac{2326014617}{19269114} & \frac{49388481}{115614684} \end{bmatrix} \begin{bmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} + h \begin{bmatrix} 0 & \frac{9603792}{113881313} & 0 & 0 \\ 0 & 0 & \frac{19789614}{1972285} & 0 \\ 0 & 0 & 0 & \frac{845710}{46087} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix} + h$$
$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{951570371}{3211519} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{bmatrix} \qquad (24)$$

(24) can be transform into matrix form

$$A_0 Y_m = A_1 Y_{m-1} + h(B_0 F_{m-1} + B_1 F_m) \qquad (25)$$

Where

$$A_0 = \begin{bmatrix} 1 & -\frac{6274637}{16268759} & \frac{143998979}{1708219695} & -\frac{9603792}{113881313} \\ \frac{7210474}{394457} & 1 & -\frac{21582821}{59168550} & -\frac{14016}{1972285} \\ \frac{11495780}{599131} & -\frac{6496015}{1198262} & 1 & -\frac{42690}{599131} \\ -\frac{1117145237}{19269114} & \frac{11296250177}{77076456} & -\frac{495749336}{28903671} & 1 \end{bmatrix}, \qquad (26)$$

$$A_1 = \begin{bmatrix} \frac{1298881}{341643939} & -\frac{341643939}{569406565} & \frac{72003623}{113881313} & -\frac{426060731}{341643939} \\ \frac{79696}{845265} & -\frac{41929759}{9861425} & \frac{68414023}{3944570} & -\frac{189894686}{5916855} \\ -\frac{70450}{1797393} & \frac{1295843}{1198262} & -\frac{5593225}{599131} & -\frac{676840}{105729} \\ \frac{338687}{348237} & \frac{353855969}{77076456} & -\frac{2326014617}{19269114} & \frac{49388481}{115614684} \end{bmatrix} \quad (27)$$

$$B_0 = \begin{bmatrix} 0 & \frac{9603792}{113881313} & 0 & 0 \\ 0 & 0 & \frac{19789614}{1972285} & 0 \\ 0 & 0 & 0 & \frac{845710}{46087} \\ 0 & 0 & 0 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{951570371}{3211519} & 0 & 0 & 0 \end{bmatrix} \quad 28)$$

$Y_{m-1}$ , $Y_m$ , $F_{m-1}$ and $F_m$ are column vectors defined as

$$Y_m = \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \end{bmatrix} = \begin{bmatrix} y_{3m+1} \\ y_{3m+2} \\ y_{3m+3} \\ y_{3m+4} \end{bmatrix}, Y_{m-1} = \begin{bmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} = \begin{bmatrix} y_{3(m-1)+1} \\ y_{3(m-1)+2} \\ y_{3(m-1)+3} \\ y_{3(m-1)+4} \end{bmatrix}, F_{m-1} = \begin{bmatrix} f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix} = \begin{bmatrix} f_{3(m-1)+1} \\ f_{3(m-1)+2} \\ f_{3(m-1)+3} \\ f_{3(m-1)+4} \end{bmatrix} \quad (29)$$

$$F_m = \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{bmatrix} = \begin{bmatrix} f_{3m+1} \\ f_{3m+2} \\ f_{3m+3} \\ f_{3m+4} \end{bmatrix} \quad (30)$$

Substituting scalar test equation $y' = \lambda y$ ($\lambda < 0$, $\lambda$ complex) into (25) and using $\lambda h = \bar{h}$ gives

$$A_0 Y_m = A_1 Y_{m-1} + \bar{h}(B_0 Y_{m-1} + B_1 Y_m) \quad (31)$$

The stability polynomial of (3) is obtained by evaluating

$$det[(A_0 - \bar{h}B_1)t - (A_1 + \bar{h}B_0)] = 0 \quad \text{using maple software} \quad (32)$$

To get

$$R(\bar{h}, t) = \frac{996090316807547559435103303 3}{132975325936366820357365460}h - \frac{446737709680296868675844731429}{106380260749093456285892368}t - \frac{267896832298507582085725524 9}{66487662968183410178682 73} t^4 h - \frac{193733184956304804387420096}{66487662968183410178682 73} t^3 h^2 - \frac{256081694624308812616426 08}{66487662968183410178682 73} h^2 + \frac{638171663697310966422440976921}{106380260749093456285892368} t^2 - \frac{121326420211773039356715312753 7}{199462988904550230536048190} th - \frac{567630082507188660567238515954 1}{398925977809100461072096380} t^2 h - \frac{156704338863926834777833911288 6}{332438314840917050893413 65} t^3 h + \frac{636761357124782350302383414 4}{511443561293718539836021} t^2 h^2 - \frac{283264387512266327961835113 6}{66487662968183410178682 73} th^2 + \frac{305957123435183189886671552 47}{531901303745467281429461840} - \frac{921565130889493728528082 09}{511443561293718539836021 0} t^4 - \frac{153560628738985568751170538 3}{835009895989744554834320} t^3 \quad (33)$$

By putting $\bar{h} = 0$ in (33), we obtain the first characteristic polynomial as

$$R(0, t) = -\frac{446737709680296868675844731429}{106380260749093456285892368}t + \frac{638171663697310966422440976921}{106380260749093456285892368} t^2 - \frac{153560628738985568751170538 3}{835009895989744554834320} t^3 - \frac{921565130889493728528082 09}{511443561293718539836021 0} t^4 + \frac{305957123435183189886671552 47}{531901303745467281429461840} \quad (34)$$

Since, the roots of (34) are $t_1 = 1$ $and$ $t_2, t_3, t_4 \leq 0$

Therefore, the method (3) is zero Stable.

## Consistency Conditions

**Definition 3**: A Linear Multi-Step Method is said to be consistent if its order p is greater than or equal to one. It also follows that a LMM is consistent if and only if:

$$\sum_{j=0}^{K} C_j = 0 \quad (35)$$

and

$$\sum_{j=0}^{K} jC_j = \sum_{j=0}^{K} D_j \quad (36)$$

Where $C_j$ and $D_j$ are constant coefficient matrices. It follows that (3) it is consistent if and only if $\rho(1) = 0$ and $\rho(1) = \sigma(1)$. Where $\rho$ and $\sigma$ are the 1st and 2nd characteristic polynomial respectively.

Obviously (3) has order greater than 1, that is order $p \geq 1$.

Thus, conditions (35) and (36) are met.

$$\sum_{j=0}^{7} C_j = C_0 + C_1 + C_2 + C_3 + C_4 + C_5 + C_6 + C_7$$

$$\begin{bmatrix} -\frac{1298881}{341643939} \\ -\frac{79696}{845265} \\ \frac{70450}{1797393} \\ -\frac{338687}{348237} \end{bmatrix} + \begin{bmatrix} \frac{341643939}{569406565} \\ -\frac{41929759}{9861425} \\ -\frac{1295843}{1198262} \\ -\frac{353855969}{348237} \end{bmatrix} + \begin{bmatrix} -\frac{72003623}{113881313} \\ -\frac{68414023}{3944570} \\ \frac{5593225}{599131} \\ \frac{2326014617}{19269114} \end{bmatrix} + \begin{bmatrix} \frac{426060731}{341643939} \\ \frac{1898894686}{5916855} \\ \frac{676840}{105729} \\ -\frac{49388481}{115614684} \end{bmatrix} + \begin{bmatrix} 1 \\ \frac{7210474}{394457} \\ \frac{11495780}{599131} \\ -\frac{1117145237}{19269114} \end{bmatrix} + \begin{bmatrix} -\frac{6274637}{16268759} \\ 1 \\ -\frac{6496015}{1198262} \\ \frac{11296250177}{77076456} \end{bmatrix} + \begin{bmatrix} \frac{143998979}{1708219695} \\ -\frac{21582821}{59168550} \\ 1 \\ -\frac{495749336}{28903671} \end{bmatrix}$$

$$+ \begin{bmatrix} -\frac{9603792}{113881313} \\ -\frac{14016}{1972285} \\ -\frac{42690}{599131} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (37)$$

Therefore, the condition (35) is satisfied. Also

$$\sum_{j=0}^{7} jC_j = 0 \cdot C_0 + 1 \cdot C_1 + 2 \cdot C_2 + 3 \cdot C_3 + 4 \cdot C_4 + 5 \cdot C_5 + 6 \cdot C_6 + 7 \cdot C_7$$

$$0 \cdot \begin{bmatrix} -\frac{1298881}{341643939} \\ -\frac{79696}{845265} \\ \frac{70450}{1797393} \\ -\frac{338687}{348237} \end{bmatrix} + 1 \cdot \begin{bmatrix} \frac{341643939}{569406565} \\ -\frac{41929759}{9861425} \\ -\frac{1295843}{1198262} \\ -\frac{353855969}{348237} \end{bmatrix} + 2 \cdot \begin{bmatrix} -\frac{72003623}{113881313} \\ -\frac{68414023}{3944570} \\ \frac{5593225}{599131} \\ \frac{2326014617}{19269114} \end{bmatrix} + 3 \cdot \begin{bmatrix} \frac{426060731}{341643939} \\ \frac{1898894686}{5916855} \\ \frac{676840}{105729} \\ -\frac{49388481}{115614684} \end{bmatrix} + 4 \cdot \begin{bmatrix} 1 \\ \frac{7210474}{394457} \\ \frac{11495780}{599131} \\ -\frac{1117145237}{19269114} \end{bmatrix} + 5 \cdot \begin{bmatrix} -\frac{6274637}{16268759} \\ 1 \\ -\frac{6496015}{1198262} \\ \frac{11296250177}{77076456} \end{bmatrix}$$

$$+6 \cdot \begin{bmatrix} \frac{143998979}{1708219695} \\ -\frac{21582821}{59168550} \\ 1 \\ -\frac{495749336}{28903671} \end{bmatrix} + 7 \cdot \begin{bmatrix} -\frac{9603792}{113881313} \\ -\frac{14016}{1972285} \\ -\frac{42690}{599131} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{9603792}{113881313} \\ \frac{19789614}{1972285} \\ \frac{845710}{46087} \\ \frac{951570371}{3211519} \end{bmatrix} \qquad (38)$$

And

$$\sum_{j=0}^{7} D_j = D_0 + D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{9603792}{113881313} \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{19789614}{1972285} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{845710}{46087} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{951570371}{3211519} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{9603792}{113881313} \\ \frac{19789614}{1972285} \\ \frac{845710}{46087} \\ \frac{951570371}{3211519} \end{bmatrix} \qquad (39)$$

Therefore, $\sum_{j=0}^{7} jC_j = \sum_{j=0}^{7} D_j$ .Thus, condition in (24) is also met; the method (3) is consistent.

Hence, the method (3) is Convergent in accordance with the theorem (1)

## Conclusion

The necessary and sufficient conditions for the convergence of a linear multistep method highlighted in theorem (1) are satisfied by the proposed method. The proposed scheme found to be of order 7, zero stable and consistent. The convergent method is recommended for the solution of first order system of initial value problem of ordinary differential equations.

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