International Journal of Research Publication and Reviews, Vol 4, no 1, pp 714-720, January 2023

International Journal of Research Publication and Reviews

Journal homepage: www.ijrpr.com ISSN 2582-7421

A Search on Integer Solutions to the Homogeneous Quadratic Equation with Three Unknowns $x^2 + 14y^2 = 23z^2$

***T. Mahalakshmi* [1], *P. Sowmiya* [2]**

[1]Assistant Professor, Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Trichy – 620 002, Tamil Nadu, India.

[2] PG Scholar, Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Trichy – 620 002, Tamil Nadu, India.

ABSTRACT:

This paper focuses on finding non-zero distinct integer solutions to the Homogeneous Quadratic Diophantine Equation with three unknowns given by $x^2 + 14y^2 = 23z^2$.Various sets of integer solutions are obtained. A few interesting properties among the solutions are given.

KEYWORDS: Ternary quadratic, Homogeneous quadratic, Integer solutions,

Legendre equation

2010 Mathematics Subject Classification: 11D09

INTRODUCTION:

The Ternary quadratic Diophantine equations offer an unlimited field for research because of their variety [1-3].For an extensive review of various problems, one may refer [4-19] for quadratic equations with two and three unknowns.This communication concerns with yet another interesting ternary quadratic equation $x^2 + 14y^2 = 23z^2$ representing a cone for determining its infinitely many non- zero integral solutions. Also a few interesting properties among the solutions have been presented.

METHOD OF ANALYSIS

The Ternary Quadratic Diophantine equation representing homogeneous cone under consideration is

$$x^2 + 14y^2 = 23z^2 \tag{1}$$

We present below different methods of solving (1).

METHOD I:

Equation (1) is written in the form of ratio as

$$\frac{x+3z}{z+y} = \frac{14(z-y)}{x-3z} = \frac{\alpha}{\beta}, \quad \beta \neq 0 \tag{2}$$

which is equivalent to the system of double equations

$$\beta x - \alpha y + (3\beta - \alpha)z = 0$$
$$\alpha x + 14\beta y - (14\beta + 3\alpha)z = 0$$

Applying the method of cross multiplication, the corresponding values of x, y, z satisfying (1) are given by

$$x(\alpha,\beta) = -3\alpha^2 + 14\beta^2 - 28\alpha\beta$$
$$y(\alpha,\beta) = \alpha^2 - 14\beta^2 - 6\alpha\beta$$
$$z(\alpha,\beta) = -\alpha^2 - 14\beta^2$$

Note 1:

Apart from (2), (1) is also written in the form of ratios as presented below

$$(i)\frac{x-3z}{7(z+y)}=\frac{2(z-y)}{x+3z}=\frac{\alpha}{\beta}$$

$$(ii)\frac{x-3z}{7(z+y)}=\frac{2(z+y)}{x+3z}=\frac{\alpha}{\beta}$$

$$(iii)\frac{x+3z}{2(z+y)}=\frac{7(z-y)}{x-3z}=\frac{\alpha}{\beta}$$

Following the above procedure, the solutions of (1) for choices (i), (ii), (iii) are presented below

Solutions for choice (i):

$$x(\alpha,\beta)=21\alpha^2-6\beta^2-28\alpha\beta$$
$$y(\alpha,\beta)=7\alpha^2-2\beta^2+6\alpha\beta$$
$$z(\alpha,\beta)=-7\alpha^2-2\beta^2$$

Solutions for choice (ii):

$$x(\alpha,\beta)=21\alpha^2+6\beta^2$$
$$y(\alpha,\beta)=7\alpha^2+2\beta^2$$
$$z(\alpha,\beta)=7\alpha^2+2\beta^2$$

Solutions for choice (iii):

$$x(\alpha,\beta)=6\alpha^2-21\beta^2+28\alpha\beta$$
$$y(\alpha,\beta)=-2\alpha^2+7\beta^2+6\alpha\beta$$
$$z(\alpha,\beta)=2\alpha^2+7\beta^2$$

METHOD II:

Introduction of the linear transformations

$$x=3P,\quad y=X+23T,\quad z=X+14T \tag{3}$$

in (1) leads to

$$X^2=322T^2+P^2 \tag{4}$$

which is satisfied by

In view of (3), the corresponding integer solutions to (1) are given by

$$x=966r^2-3s^2$$
$$y=322r^2+s^2+46rs$$
$$z=322r^2+s^2+28rs$$

Also, (4) is written as the system of double equations as presented below in Table 1

Table 1: System of double equations

System	1	2	3	4	5	6
$X+P$	T^2	$23T^2$	$23T$	$322T$	$161T^2$	$161T$
$X-P$	322	14	$14T$	T	2	$2T$

Solving each of the above systems, the value of X, P and T are obtained. Substituting these in (3), the corresponding solutions to (1) are found. For simplicity, we present the solutions below

Solutions for system 1:

$$x = 6K^2 - 483$$
$$y = 2K^2 + 46K + 161$$
$$z = 2K^2 + 28K + 161$$

Solutions for system 2:

$$x = 138K^2 - 21$$
$$y = 46K^2 + 46K + 7$$
$$z = 46K^2 + 28K + 7$$

Solutions for system 3:

$$x = 27K$$
$$y = 83K$$
$$z = 65K$$

Solutions for system 4:

$$x = 963K$$
$$y = 369K$$
$$z = 351K$$

Solutions for system 5:

$$x = 966K^2 - 3$$
$$y = 322K^2 + 46K + 1$$
$$z = 322K^2 + 28K + 1$$

Solutions for system 6:

$$x = 477K$$
$$y = 209K$$
$$z = 191K$$

METHOD III:

Assume

$$z(a,b) = a^2 + 14b^2 \tag{5}$$

Case (i):

Write 23 as

$$23 = \left(3 + i\sqrt{14}\right)\left(3 - i\sqrt{14}\right) \tag{6}$$

Using (5) and (6) in (1) and employing the method of factorization, consider

$$x + i\sqrt{14}y = (3 + i\sqrt{14})(a + i\sqrt{14}b)^2$$

Equating real and imaginary parts, we have

$$x = 3a^2 - 42b^2 - 28ab$$
$$y = a^2 - 14b^2 - 6ab$$
$$z = a^2 + 14b^2$$

Case(ii):

Write 23 as

$$23 = \frac{(43 + i\sqrt{14})(43 - i\sqrt{14})}{9^2} \tag{7}$$

Using (5) and (7) in (1) and employing the method of factorization, consider

$$x + i\sqrt{14}y = \frac{(43 + i\sqrt{14})}{9}(a + i\sqrt{14}b)^2$$

Equating real and imaginary parts and replacing a by 3A, b by 3B, we have

$$\left.\begin{aligned} x(A,B) &= 43A^2 - 602B^2 - 28AB \\ y(A,B) &= A^2 - 14B^2 + 86AB \end{aligned}\right\} \tag{8}$$

and from we have

$$z(A,B) = 9A^2 + 126B^2 \tag{9}$$

Thus (8) and (9) represent the integer solutions to (1).

METHOD IV:

Equation (1) is written as

$$x^2 + 14y^2 = 23z^2 * 1 \tag{10}$$

Write 1 as

$$1 = \frac{(1+i4\sqrt{14})(1-i4\sqrt{14})}{225} \tag{11}$$

Substituting (5), (6) and (11) in (10) and following the procedure as above and replacing a by 15A and b by 15B, the corresponding solutions to (1) are given by

$$x(A,B) = -795A^2 + 1113B^2 - 5460AB$$
$$y(A,B) = 195A^2 - 2730B^2 - 1590AB$$
$$z(A,B) = 225A^2 + 3150B^2$$

Note 2:

It is seen that 1 is also represented as follows

$$(iv) \quad 1 = \frac{\left(14r^2 - s^2 + i\sqrt{14}\,2rs\right)\left(14r^2 - s^2 - i\sqrt{14}\,2rs\right)}{\left(14r^2 + s^2\right)^2}$$

Following the above procedure, the solutions of (1) are obtained.

METHOD V:

Write (1) as

$$23z^2 - 14y^2 = x^2 * 1 \tag{12}$$

Let

$$x = 23a^2 - 14b^2 \tag{13}$$

Consider 1 as

$$1 = \frac{\left(\sqrt{23}+\sqrt{14}\right)\left(\sqrt{23}-\sqrt{14}\right)}{9} \tag{14}$$

Using (13) & (14) in (12) and employing the method of factorization,consider

$$\sqrt{23}z + \sqrt{14}y = \frac{1}{3}\left(\sqrt{23}+\sqrt{14}\right)\left(\sqrt{23}a + \sqrt{14}b\right)^2$$

Equating the coefficients of corresponding terms, we have

$$\left.\begin{aligned} z &= \frac{1}{3}\left(23a^2 + 14b^2 + 28ab\right) \\ y &= \frac{1}{3}\left(23a^2 + 14b^2 + 46ab\right) \end{aligned}\right\} \tag{15}$$

Replacing a by 3A and b by 3B in (13) & (15) the corresponding integer

$$\left.\begin{aligned} x &= 207A^2 - 126B^2 \\ y &= 69A^2 + 42B^2 + 138AB \\ z &= 69A^2 + 42B^2 + 84AB \end{aligned}\right\} \quad (16)$$

Then (16) gives the integer solution to (1).

METHOD VI:

Consider (1) as

$$23z^2 - x^2 = 14y^2 \quad (17)$$

Let

$$y = 23a^2 - b^2 \quad (18)$$

Write 14 as

$$14 = (\sqrt{23}+3)(\sqrt{23}-3) \quad (19)$$

Using (18) & (19) in (17) and employing the method of factorization, consider

$$(\sqrt{23}z + x) = (\sqrt{23}+3)(\sqrt{23}a + b)^2 \quad (20)$$

Equating the coefficients of corresponding terms, we have

$$\left.\begin{aligned} x &= 69a^2 + 3b^2 + 46ab \\ z &= 23a^2 + b^2 + 6ab \end{aligned}\right\} \quad (21)$$

Then, (18) & (21) gives the integer solution of (1).

GENERATION OF INTEGER SOLUTIONS

Let (x_0, y_0, z_0) be any given integer solution to (1). We illustrate below the method of obtaining a general formula for generating sequence of integer solutions based on the given solution.

Case (i)

Let

$$\begin{aligned} x_1 &= -x_0 + 5h \\ y_1 &= y_0 \qquad h \neq 0 \\ z_1 &= z_0 + h \end{aligned} \quad (22)$$

be the second solution of (1). Substituting (22) in (1) & performing a few calculations, we have

$$h = 5x_0 + 23z_0$$

and then

$$\begin{aligned} x_1 &= 24x_0 + 115z_0 \\ z_1 &= 5x_0 + 24z_0 \end{aligned}$$

This is written in the form of matrix as

$$\begin{pmatrix} x_1 \\ z_1 \end{pmatrix} = M \begin{pmatrix} x_0 \\ z_0 \end{pmatrix} \quad (23)$$

where

$$M = \begin{pmatrix} 24 & 115 \\ 5 & 24 \end{pmatrix}$$

Repeating the above process, the general solution (x_n, z_n) to (1) is given by

$$\begin{pmatrix} x_n \\ z_n \end{pmatrix} = M^n \begin{pmatrix} x_0 \\ z_0 \end{pmatrix}$$

To find M^n, the eigen values of M are $\alpha = 24+5\sqrt{23}$, $\beta = 24-5\sqrt{23}.$

We know that $M^n = \frac{\alpha^n}{(\alpha-\beta)}(M-\beta I)+\frac{\beta^n}{(\beta-\alpha)}(M-\alpha I)$

Using the above formula, we have

$$M^n = \begin{pmatrix} \frac{\alpha^n+\beta^n}{2} & \frac{\sqrt{23}(\alpha^n-\beta^n)}{2} \\ \frac{\alpha^n-\beta^n}{2\sqrt{23}} & \frac{\alpha^n+\beta^n}{2} \end{pmatrix}$$

Thus the general solution (x_n, y_n, z_n) to (1) is given by

Case (ii)

Let

$$x_1 = 9x_0$$
$$y_1 = h+9y_0$$
$$z_1 = h-9z_0 \quad , h \neq 0$$

Repeating the process as in the case (i) the corresponding general solution (x_n, y_n, z_n) to (1) is given by

Case (iii)

Let

$$x_1 = h-15x_0$$
$$y_1 = h-15y_0$$
$$z_1 = 15z_0 \quad , h \neq 0$$

Repeating the process as in the case (i) the corresponding general solution (x_n, y_n, z_n) to (1) is given by

CONCLUSION

In this paper, we have made an attempt to obtain all integer solutions to (1). As (1) is symmetric in x, y, z it is to be noted that, if (x, y, z) is any positive integer solution to (1),then the triples (-x, y, z), (x,-y, z), (x, y,-z), (x, -y, -z), (-x, y, -z), (-x, -y, z) (-x, -y, -z) also satisfy (1). To conclude, one may search for integer solutions to other choices of homogeneous cones along with suitable properties.

REFERENCES:

[1]. L.E. Dickson, History of Theory of Numbers, vol 2, Chelsea publishing company, New York, (1952).

[2]. L.J. Mordell, Diophantine Equations, Academic press, London, (1969).

[3]. R.D. Carmichael, The theory of numbers and Diophantine analysis, New York, Dover, (1959).

[4]. M.A. Gopalan, S. Vidhyalakshmi, A. Kavitha and D. Marymadona, On the Ternary Quadratic Diophantine equation $3(x^2+y^2)-2xy=4z^2$, International Journal of Engineering science and Management, 5(2) (2015) 11-18.

[5]. K. Meena, S. Vidhyalakshmi, E. Bhuvaneshwari and R. Presenna, On ternary quadratic Diophantine equation $5(X^2+Y^2)-6XY=20Z^2$, International Journal of Advanced Scientific Research, 1(2) (2016) 59-61.

[6] S. Devibala and M.A. Gopalan, On the ternary quadratic Diophantine Equation $7x^2+y^2=z^2$, International Journal of Emerging Technologies in Engineering Research, 4(9) (2016).

[7] N. Bharathi, S. Vidhyalakshmi, Observation on the Non-Homogeneous Ternary Quadratic Equation $x^2-xy+y^2+2(x+y)+4=12z^2$, Journal of mathematics and informatics, vol.10, 2017, 135-140.

[8] A. Priya, S. Vidhyalakshmi, On the Non-Homogeneous Ternary Quadratic Equation $2(x^2+y^2)-3xy+(x+y)+1=z^2$, Journal of mathematics and informatics, vol.10, 2017, 49-55.

[9] M.A. Gopalan, S. Vidhyalakshmi and U.K. Rajalakshmi, On ternary quadratic Diophantine equation $5(x^2+y^2)-6xy=196z^2$, Journal of mathematics, 3(5) (2017) 1-10.

[10] M.A. Gopalan, S. Vidhyalakshmi and S. Aarthy Thangam, On ternary quadratic Equation $x(x+y)=z+20$, IJIRSET, 6(8) (2017) 15739-15741.

[11] M.A. Gopalan and Sharadha Kumar, "On the Hyperbola $2x^2-3y^2=23$", Journal of Mathematics and Informatics, vol-10, Dec (2017), 1-9.

[12] T.R. Usha Rani and K.Ambika, Observation on the Non-Homogeneous Binary Quadratic Diophantine Equation $5x^2-6y^2=5$, Journal of Mathematics and Informatics, vol-10, Dec (2017), 67-74.

[13] T.R. Usha Rani, V. Bahavathi, S. Sridevi, "Observations on the Non-homogeneous binary Quadratic Equation $8x^2-3y^2=20$ ", IRJET, volume: 06, Issue: 03, 2019, 2375-2382.

[14] S. Vidhyalakshmi, T. Mahalakshmi, M.A. Gopalan, A study on the Non-Homogeneous Ternary Quadratic Diophantine Equation $4(x^2+y^2)-7xy+x+y+1=31z^2$, International journal of Advance in Engineering and Management (IJAEM), vol 2, Issue: 01, June(2020), 55-58.

[15] S. Vidhyalakshmi, T. Mahalakshmi, M.A. Gopalan and S. Shanthia, Observations On The Homogeneous Ternary Quadratic Diophantine Equation With Three Unknowns $y^2+5x^2=21z^2$, Journal of Information and computational science, vol 10, Issue: 3, March(2020), 822-831

[16] M.A. Gopalan, S. Vidhyalakshmi, J. Shanthi, V. Anbuvalli, On Finding the integer Solutions of Ternary Quadratic Diophantine Equation $3(x^2+y^2)-5xy=36z^2$, International journal of Precious Engineering Research and Applications (IJPERA), vol 7, Issue: 1, May(2022), 34-38.

[17] S. Vidhyalakshmi, M.A. Gopalan, On Finding Integer Solutions of Ternary Quadratic Equation $x^2+y^2=z^2-5$, Purakala UGC Care Approved Journal, vol 31, Issue: 2, May(2022), 920-926.

[18] S. Vidhyalakshmi, M.A. Gopalan, On Finding Integer Solutions of Ternary Quadratic Equation $x^2+y^2=z^2-12$, International Journal of Research Publication and Reviews, vol 3,Issue: 8, August(2022), 2146-2155.

[19] S. Vidhyalakshmi, M.A. Gopalan, On Finding Integer Solutions of Ternary Quadratic Equation $x^2+y^2=z^2-10$, International Journal of Progressive Research in Engineering Management and Science (IJPREMS), vol 02, Issue: 09, September(2022), 58-60.