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On Finding Integer Solutions to the Homogeneous Ternary Quadratic Diophantine Equation 2 ($X^2 + Y^2$) – 3XY = $32Z^2$

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Abstract:

This paper concerns with the problem of obtaining non-zero distinct integer solutions to the homogeneous quadratic Diophantine equation with three unknowns given by 2 ($x^2 + y^2$) – 3xy = $32z^2$.Various sets of integer solutions are obtained. A few interesting properties among the solutions are given.

Keywords: Ternary quadratic, Homogeneous quadratic, Integer solutions, Legendre equation

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INTRODUCTION

The ternary quadratic Diophantine equation offers an unlimited field for research due to their variety [1-3]. In particular, one may refer [4-19] for quadratic equations with two and three unknowns. This communication concerns with yet another interesting ternary quadratic equation 2 ($x^2 + y^2$) – 3xy = $32z^2$ is analysed for its non-zero distinct integer solutions through different methods.

METHODS OF ANALYSIS

The ternary quadratic Diophantine equation to be solved for -zero distinct integral solution is

$$2\left(x^2+y^2\right)-3xy=32z^2 \tag{1}$$

Introduction of the linear transformations

$$x=u+v,\quad y=u-v,\quad u\neq v\neq 0 \tag{2}$$

in (1) leads to

$$u^2+7v^2=32z^2 \tag{3}$$

The above equation is solved for u, v and z through different methods and using (2), the values of x and ysatisfying (1), are obtained which are illustrated below

Method I:

Write (3) in the form of ratio as

$$\frac{u+5z}{z+v}=\frac{7(z-v)}{u-5z}=\frac{\alpha}{\beta},\quad \beta\neq 0 \tag{4}$$

which is equivalent to the system of double equations

$$\beta u-\alpha v+(5\beta-\alpha)z=0$$
$$-\alpha u-7\beta v+(7\beta+5\alpha)z=0$$

Solving the above system of double equations and using (2), the corresponding integer solutions to (1) are found to be

$$x = 4\alpha^2 - 28\beta^2 + 24\alpha\beta$$
$$y = 6\alpha^2 - 42\beta^2 + 4\alpha\beta$$
$$z = \alpha^2 + 7\beta^2$$

Note 1:

It is noted that (3) may also be written in the form of ratios as below

(i) $$\frac{u+5z}{7(z-v)} = \frac{z+v}{u-5z} = \frac{\alpha}{\beta}$$

(ii) $$\frac{u-5z}{z+v} = \frac{7(z-v)}{u+5z} = \frac{\alpha}{\beta}$$

(iii) $$\frac{u-5z}{7(z-v)} = \frac{z+v}{u+5v} = \frac{\alpha}{\beta}$$

(iv) $$\frac{u-5z}{z-v} = \frac{7(z+v)}{u+5z} = \frac{\alpha}{\beta}$$

For each of the above ratios, the corresponding integer solutions to (1) are exhibited below

Solutions obtained through (i)

$$x = 42\alpha^2 + 6\beta^2 - 4\alpha\beta$$
$$y = 28\alpha^2 + 4\beta^2 - 24\alpha\beta$$
$$z = -7\alpha^2 - \beta^2$$

Solutions obtained through (ii)

$$x = -6\alpha^2 + 42\beta^2 + 4\alpha\beta$$
$$y = -4\alpha^2 + 28\beta^2 + 24\alpha\beta$$
$$z = \alpha^2 - 7\beta^2$$

Solutions obtained through (iii)

$$x = 28\alpha^2 - 4\beta^2 - 24\alpha\beta$$
$$y = 42\alpha^2 - 6\beta^2 - 4\alpha\beta$$
$$z = -7\alpha^2 - \beta^2$$

Solution obtained through (iv)

$$x = 4\alpha^2 - 28\beta^2 - 24\alpha\beta$$
$$y = 6\alpha^2 - 42\beta^2 - 4\alpha\beta$$
$$z = -\alpha^2 - 7\beta^2$$

Method II:

Introducing the linear transformations

$$Z = X + 7T, \quad v = X + 32T, \quad u = 5P \tag{5}$$

in (3), it gives

$$X^2 = 224T^2 + P^2 \tag{6}$$

which is satisfied by

$$\left.\begin{aligned} T &= 2rs \\ P &= 224r^2 - s^2 \\ X &= 224r^2 + s^2 \end{aligned}\right\} \tag{7}$$

From (7), (5) & (2), we obtain the integer solutions to (1) as given below

$$x = 1344r^2 - 4s^2 + 64rs$$
$$y = 896r^2 - 6s^2 - 64rs$$
$$z = 224r^2 + s^2 + 14rs$$

It is to be noted that (6) may be represented as the system of double equation as shown in Table: 1

Table: 1 System of double equations

System	1	2	3	4	5
$X+P$	T^2	$7T^2$	$2T^2$	$4T^2$	$8T^2$
$X-P$	224	32	112	56	28

System	**6**	**7**	**8**	**9**	**10**
$X+P$	$16T^2$	$14T^2$	$28T^2$	$56T^2$	$112T^2$
$X-P$	14	16	8	4	2

System	**11**	**12**	**13**	**14**	**15**
$X+P$	$32T$	$224T$	T	$2T$	$14T$
$X-P$	$7T$	T	$224T$	$112T$	$16T$

Solving each of the system of double equations in Table: 1, the values of X, P & T are obtained, from (5) & (2), the corresponding solutions to (1) are found and they are exhibited below

Solutions from system 1

$$x = 12k^2 + 64k - 448$$
$$y = 8k^2 - 64k - 672$$
$$z = 2k^2 + 14k + 112$$

Solutions from system 2

$$x = 84k^2 + 64k - 64$$
$$y = 56k^2 - 64k - 96$$
$$z = 14k^2 + 14k + 16$$

Solutions from system 3

$$x = 24k^2 + 64k - 224$$
$$y = 16k^2 - 64k - 336$$
$$z = 4k^2 + 14k + 56$$

Solutions from system 4

$$x = 48k^2 + 64k - 112$$
$$y = 32k^2 - 64k - 168$$
$$z = 8k^2 + 14k + 28$$

Solutions from system 5

$$x = 96k^2 + 64k - 56$$
$$y = 64k^2 - 64k - 84$$
$$z = 16k^2 + 14k + 14$$

Solutions from system 6

$$x = 192k^2 + 64k - 28$$
$$y = 128k^2 - 64k - 42$$
$$z = 32k^2 + 14k + 7$$

Solutions from system 7

$$x = 168k^2 + 64k - 32$$
$$y = 112k^2 - 64k - 48$$
$$z = 28k^2 + 14k + 8$$

Solutions from system 8

$$x = 336k^2 + 64k - 16$$
$$y = 224k^2 - 64k - 24$$
$$z = 56k^2 + 14k + 4$$

Solutions from system 9

$$x = 672k^2 + 64k - 8$$
$$y = 448k^2 - 64k - 12$$
$$z = 112k^2 + 14k + 2$$

Solutions from system 10

$$x = 1344k^2 + 64k - 4$$
$$y = 896k^2 - 64k - 6$$
$$z = 224k^2 + 14k + 1$$

Solutions from system 11

$$x = 228k$$
$$y = 22k$$
$$z = 53k$$

Solutions from system 12

$$x = 1404k$$
$$y = 826k$$
$$z = 239k$$

Solutions from system 13

$$x = -826k$$
$$y = -1404k$$
$$z = 239k$$

Solutions from system 14

$$x = -372k$$
$$y = -728k$$
$$z = 128k$$

Solutions from system 15

$$x = 84k$$
$$y = -104k$$
$$z = 44k$$

Method III:

Assume

$$z = a^2 + b^2 \tag{8}$$

Case (i):

Write 32 as

$$32 = (5 + i\sqrt{7})(5 - i\sqrt{7}) \tag{9}$$

Using (8) and (9) in (3) and employing the method of factorization, define

$$(u+i\sqrt{7}v)=(5+i\sqrt{7})(a+i\sqrt{7}b)^2$$

Equating the real and imaginary parts, we get

$$u=5a^2-35b^2-14ab$$
$$v=a^2-7b^2+10ab$$

in view of (2), we obtain

$$\left.\begin{aligned} x&=6a^2-42b^2-4ab \\ y&=4a^2-28b^2-24ab \end{aligned}\right\} \quad (10)$$

Thus (8) and (10) represent the integer solution to (1).

Case (ii):

One can write 32 as

$$32=\frac{(13+i7\sqrt{7})(13-i7\sqrt{7})}{4^2} \quad (11)$$

Using (8) and (11) in (3) and applying the method of factorization, define

$$(u+i\sqrt{7}v)=\frac{(13+i7\sqrt{7})}{4}(a+i\sqrt{7}b)^2$$

Equating the real and imaginary parts, we get

$$u=\frac{13a^2-91b^2-98ab}{4}$$
$$v=\frac{7a^2-49b^2+26ab}{4}$$

in view of (2), we obtain

$$\left.\begin{aligned} x&=\frac{20a^2-140b^2-72ab}{4} \\ y&=\frac{6a^2-42b^2-124ab}{4} \end{aligned}\right\} \quad (12)$$

To obtain the integer solutions, replacing a by 2A and b by 2B in (8) & (12), the corresponding integer solutions of (1) are given by

$$\left.\begin{aligned} x&=20A^2-140B^2-72AB \\ y&=6A^2-42B^2-124AB \\ z&=4A^2+28B^2 \end{aligned}\right\} \quad (13)$$

Method IV:

Equation (3) can be written as

$$u^2+7v^2=32z^2*1 \quad (14)$$

Write 1 on the R.H.S. of (14) as

$$1=\frac{(1+i3\sqrt{7})(1-i3\sqrt{7})}{8^2} \quad (15)$$

Using (8), (9) & (15) in (14) and utilizing the method of factorization, define

$$(u+i\sqrt{7}v)=(5+i\sqrt{7})(a+i\sqrt{7}b)^2\left[\frac{(1+i3\sqrt{7})}{8}\right]$$

Equating the real and imaginary parts, the values of u and v are obtained as

$$u=\frac{-16a^2+112b^2-224ab}{8}$$
$$v=\frac{16a^2-112b^2-32ab}{8}$$

Proceeding as in case (ii), we get

$$\left.\begin{array}{l} x=-128AB \\ y=-16A^2+112B^2-96AB \\ Z=4A^2+28B^2 \end{array}\right\} \quad (16)$$

Thus (16) represent the non-zero distinct solution of (1)

Note 2:

It is seen that 1 is also represented as follows

$$(v) \quad 1=\frac{(3+i\sqrt{7})(3-i\sqrt{7})}{4^2}$$

$$(vi) \quad 1=\frac{(3+i4\sqrt{7})(3-i4\sqrt{7})}{11^2}$$

$$(vii) \quad 1=\frac{(7r^2-s^2+i\sqrt{7}2rs)(7r^2-s^2-i\sqrt{7}2rs)}{(7r^2+s^2)^2}$$

Following the above procedure, the solutions of (1) are obtained.

Method V:

Consider (3) as

$$32z^2-7v^2=u^2*1 \quad (17)$$

Let

$$u=32a^2-7b^2 \quad (18)$$

Consider 1 as

$$1=\frac{\left(\sqrt{32}+\sqrt{7}\right)\left(\sqrt{32}-\sqrt{7}\right)}{25} \quad (19)$$

Using (18) & (19) in (17) and employing the method of factorization, consider

$$\sqrt{32}z+\sqrt{7}v=\frac{1}{5}\left(\sqrt{32}+\sqrt{7}\right)\left(\sqrt{32}a+\sqrt{7}b\right)^2$$

Equating the coefficients of corresponding terms, we have

$$\left.\begin{array}{l} z=\frac{1}{5}\left(32a^2+7b^2+14ab\right) \\ v=\frac{1}{5}\left(32a^2+7b^2+64ab\right) \end{array}\right\} \quad (20)$$

Replacing a by 5A, b by 5B in (18) & (20) the corresponding integer solutions to (17) are given by

$$\left.\begin{array}{l} u=800A^2-175B^2 \\ v=160A^2+35B^2+320AB \end{array}\right\} \quad (21)$$

$$z=160A^2+35B^2+70AB \quad (22)$$

Substituting (21) in (2), we have

$$\left.\begin{array}{l} x=960A^2-140B^2+320AB \\ y=640A^2-210B^2-320AB \end{array}\right\} \quad (23)$$

Then (22) & (23) give the integer solution to (1).

Method VI:

Consider (3) as

$$32z^2-u^2=7v^2 \quad (24)$$

Let

$$v=32a^2-b^2 \quad (25)$$

Write 7 as

$$7=\left(\sqrt{32}+5\right)\left(\sqrt{32}-5\right) \tag{26}$$

Using (25) & (26) in (24) and employing the method of factorization, consider

$$\left(\sqrt{32}z+u\right)=\left(\sqrt{32}+5\right)\left(\sqrt{32}a+b\right)^2 \tag{27}$$

Equating the coefficients of corresponding terms, we have

$$z=32a^2+b^2+10ab \tag{28}$$

$$u=160a^2+5b^2+64ab \tag{29}$$

From (25) & (29) in (2), we have

$$\left.\begin{aligned} x&=192a^2+4b^2+64ab \\ y&=128a^2+6b^2+64ab \end{aligned}\right\} \tag{30}$$

Then, (28) & (30) gives the integer solution of (1).

Generation of Integer Solutions

Let (u_0, v_0, z_0) be any given integer solution to (3). We illustrate below the method of obtaining a general formula for generating sequence of integer solutions based on the given solution.

Case (i)

Let

$$\begin{aligned} u_1&=-u_0+6h \\ v_1&=v_0 \qquad\qquad h\neq 0 \\ z_1&=z_0+h \end{aligned} \tag{31}$$

be the second solution of (3). Substituting (31) in (3) & performing a few calculations, we have

$$h=3u_0+16z_0$$

and then

$$u_1=17u_0+96z_0$$

$$z_1=3u_0+17z_0$$

This is written in the form of matrix as

$$\begin{pmatrix} u_1 \\ z_1 \end{pmatrix}=M\begin{pmatrix} u_0 \\ z_0 \end{pmatrix} \tag{32}$$

where $M=\begin{pmatrix} 17 & 96 \\ 3 & 17 \end{pmatrix}$

Repeating the above process, the general solution (u_n, z_n) to (3) is given by

$$\begin{pmatrix} u_n \\ z_n \end{pmatrix}=M^n\begin{pmatrix} u_0 \\ z_0 \end{pmatrix}$$

To find M^n, the eigen values of M are $\alpha=17+12\sqrt{2}$, $\beta=17-12\sqrt{2}.$

We know that $M^n=\dfrac{\alpha^n}{(\alpha-\beta)}(M-\beta I)+\dfrac{\beta^n}{(\beta-\alpha)}(M-\alpha I)$

Using the above formula, we have

$$M^n=\begin{pmatrix} \dfrac{\alpha^n+\beta^n}{2} & \dfrac{\sqrt{23}\left(\alpha^n-\beta^n\right)}{2} \\ \dfrac{\alpha^n-\beta^n}{2\sqrt{23}} & \dfrac{\alpha^n+\beta^n}{2} \end{pmatrix}$$

Thus the general solution (u_n, v_n, z_n) to (3) is given by

$$v_n = v_0$$

$$z_n = \left(\frac{\alpha^n - \beta^n}{8\sqrt{2}}\right)u_0 + \left(\frac{\alpha^n + \beta^n}{2}\right)z_0$$

From (2) we have,

Thus the general solution (x_n, y_n, z_n) to (1) is given by

Case (ii)

Let

$$u_1 = u_0$$
$$v_1 = v_0 + 2h$$
$$z_1 = -z_0 + h \quad , h \neq 0$$

Repeating the process as in the case (i) the corresponding general solution (x_n, y_n, z_n) to (1) is given by

Case (iii)

Let

$$u_1 = -8u_0 + h$$
$$v_1 = -8v_0 + h$$
$$z_1 = 8z_0 \quad , h \neq 0$$

Repeating the process as in the case (i) the corresponding general solution (x_n, y_n, z_n) to (1) is given by

CONCLUSION:

In this paper, we have presented four different methods of obtaining infinitely many non-zero distinct integer solutions of the homogeneous cone given by $2(X^2+Y^2) - 3XY = 32Z^2$. To conclude, one may search for other patterns of solutions and their corresponding properties.

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