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On Finding Integral Solutions of Ternary Quadratic Equation

$$x^2 + y^2 = z^2 - 12$$

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Abstract:

This paper illustrates the process of obtaining different sets of non-zero distinct integer solutions to the non-homogeneous ternary quadratic Diophantine equations given by $x^2 + y^2 = z^2 - 12$.

Keywords: non-homogeneous quadratic , ternary quadratic, integer solutions

Introduction:

It is known that Diophantine equations with multidegree and multiple variables are rich in variety[1,2].

While searching for the collection of second degree equations with three unknowns,the authors came across the papers [3-13]in which the authors obtained integer solutions to the ternary quadratic equations $x^2 + y^2 = z^2 + N, N = 1, \pm 4, \pm 8, 12, -2k^2, 10$ The above papers motivated us for obtaining non zero distinct integer solutions to the above equation for other values to N. This communication illustrates process of obtaining different sets of non-zero distinct integer solutions to the non-homogeneous ternary quadratic Diophantine equation given by $x^2 + y^2 = z^2 - 12$.

Method of analysis:

The non-homogeneous ternary quadratic Diophantine equation under consideration is

$$x^2 + y^2 = z^2 - 12 \quad (1)$$

The process of obtaining different sets of integer solutions to (1) is illustrated below:

Illustration 1:

The choice

$$z = x + h, h \geq 0 \quad (2)$$

in (1) leads to the parabola

$$y^2 = 2hx + h^2 - 12 \quad (3)$$

It is possible to choose h, x so that the R.H.S. of (3) is a perfect square and the value of y is obtained. Substituting the values of h, x in (2),the corresponding value of z satisfying (1) is obtained. For simplicity and brevity ,a few examples are given in Table 1 below:

Table 1 : Examples

h	x	y	z
1	$2k^2 \pm 2k + 6$	$2k \pm 1$	$3k^2 \pm 2k + 7$
2	$k^2 + 2$	$2k$	$k^2 + 4$
3	$\frac{3^{2s} + 3}{6}$	3^s	$\frac{3^{2s} + 21}{6}$
4	$2k^2 \pm 2k$	$4k \pm 2$	$2k^2 \pm 2k + 4$
6	$3k^2 - 2$	$6k$	$3k^2 + 4$

Illustration 2:

The substitution of the linear transformations

$$z = (u+1)s \ , x = us \quad (4)$$

in (1) leads to the negative pell equation

$$y^2 = (2u+1)s^2 - 12 \quad (5)$$

for which the integer solutions exist when u takes particular values.

Example :1

Considering the value of u to be 1 in (4),it gives the negative pell equation

$$y^2 = 3s^2 - 12 \tag{6}$$

After some algebra ,the corresponding integer solutions to (6) are given by

$$y_{n+1} = (3f_n + 2\sqrt{3}g_n) \tag{7}$$

$$s_{n+1} = (2f_n + \sqrt{3}g_n) \tag{8}$$

where $f_n = (2+\sqrt{3})^{n+1} + (2-\sqrt{3})^{n+1}$, $g_n = (2+\sqrt{3})^{n+1} - (2-\sqrt{3})^{n+1}$

Using (8) in (4), one obtains that

$$x_{n+1} = (2f_n + \sqrt{3}g_n) \, , z_{n+1} = (4f_n + 2\sqrt{3}g_n) \tag{9}$$

Thus,(7) and (9) represent the integer solutions to (1).

Example :2

Considering the value of u to be 3 in (4),it gives the negative pell equation

$$y^2 = 7s^2 - 12 \tag{10}$$

After some algebra ,the corresponding integer solutions to (10) are given by

$$y_{n+1} = (2f_n + \sqrt{7}g_n) \tag{11}$$

$$s_{n+1} = (f_n + \frac{2g_n}{\sqrt{7}}) \tag{12}$$

where $f_n = (8+3\sqrt{7})^{n+1} + (8-3\sqrt{7})^{n+1}$, $g_n = (8+3\sqrt{7})^{n+1} - (8-3\sqrt{7})^{n+1}$

Using (12) in (4), one obtains that

$$x_{n+1} = (3f_n + \frac{6g_n}{\sqrt{7}}) \, , z_{n+1} = (4f_n + \frac{8g_n}{\sqrt{7}}) \tag{13}$$

Thus,(11) and (13) represent the integer solutions to (1).

Illustration 3:

The substitution of the linear transformations

$$z=(u+3)s \ , x=us \tag{14}$$

in (1) leads to the negative pell equation

$$y^2=(6u+9)s^2-12 \tag{15}$$

for which the integer solutions exist when u takes particular values.

Example :3

Considering the value of u to be 2 in (14),it gives the negative pell equation

$$y^2=21s^2-12 \tag{16}$$

After some algebra ,the corresponding integer solutions to (16) are given by

$$y_{n+1}=\frac{1}{2}(3f_n+\sqrt{21}g_n) \tag{17}$$

$$s_{n+1}=\frac{1}{2}(f_n+\frac{3g_n}{\sqrt{21}}) \tag{18}$$

where $f_n=(55+12\sqrt{21})^{n+1}+(55-12\sqrt{21})^{n+1}$, $g_n=(55+12\sqrt{21})^{n+1}-(55-12\sqrt{21})^{n+1}$

Using (18) in (14), one obtains that

$$x_{n+1}=(f_n+\frac{3g_n}{\sqrt{21}}) \ , z_{n+1}=\frac{5}{2}(f_n+\frac{3g_n}{\sqrt{21}}) \tag{19}$$

Thus,(17) and (19) represent the integer solutions to (1).

Example :4

Considering the value of u to be 14 in (14),it gives the negative pell equation

$$y^2=93s^2-12 \tag{20}$$

After some algebra ,the corresponding integer solutions to (6) are given by

$$y_{n+1} = \frac{1}{2}(9f_n + \sqrt{93}\,g_n) \quad (21)$$

$$s_{n+1} = \frac{1}{2}(f_n + \frac{9g_n}{\sqrt{93}}) \quad (22)$$

where

$$f_n = (12151 + 1260\sqrt{93})^{n+1} + (12151 - 1260\sqrt{93})^{n+1}\ ,\ g_n = (12151 + 1260\sqrt{93})^{n+1} - (12151 - 1260\sqrt{93})^{n+1}$$

Using (22) in (14), one obtains that

$$x_{n+1} = (7f_n + \frac{63g_n}{\sqrt{93}})\ ,\ z_{n+1} = \frac{17}{2}(f_n + \frac{9g_n}{\sqrt{93}}) \quad (23)$$

Thus,(21) and (23) represent the integer solutions to (1).

Illustration 4:

The substitution of the linear transformations

$$z = u + v\ ,\ x = u - v\ ,\ u \neq v \neq 0 \quad (24)$$

in (1) leads to

$$y^2 = 4(uv - 3) \quad (25)$$

Remember that u, v are non-zero distinct integers and it is possible to choose them

such that the R.H.S. of (25) is a perfect square and the value of y is obtained. Substituting the values of u, v in (24), the corresponding values of x, y are found . A few numerical examples are exhibited in Table 1: below:

Table 1: Examples

v	u	x	y	z
1	s^2+3	s^2+2	$2s$	s^2+4
2	$2s^2+2s+2$	$2s^2+2s$	$4s+2$	$2s^2+2s+4$

3	$3s^2+1$	$3s^2-2$	$6s$	$3s^2+4$
4	s^2+s+1	s^2+s-3	$4s+2$	s^2+s+5
6	$6s^2-6s+2$	$6s^2-6s-4$	$12s-6$	$6s^2-6s+8$
7	$7s^2-4s+1$ $7s^2-10s+4$	$7s^2-4s-6$ $7s^2-10s-3$	$14s-4$ $14s-10$	$7s^2-4s+8$ $7s^2-10s+11$

Illustration 5:

The substitution of the linear transformations

$$z = kx, k \neq 1 \tag{26}$$

in (1) leads to the negative pell equation

$$y^2 = (k^2-1)\, x^2 - 12 \tag{27}$$

for which the integer solutions exist when k takes particular values.

Example :5

Considering the value of k to be 2 in (26),it gives the negative pell equation

$$y^2 = 3x^2 - 12 \tag{28}$$

After some algebra ,the corresponding integer solutions to (28) are given by

$$y_{n+1} = (3f_n + 2\sqrt{3}\,g_n) \tag{29}$$

$$x_{n+1} = (2f_n + \sqrt{3}\,g_n) \tag{30}$$

where

$$f_n = (2+\sqrt{3})^{n+1} + (2-\sqrt{3})^{n+1}\ , g_n = (2+\sqrt{3})^{n+1} - (2-\sqrt{3})^{n+1}$$

Using (30) in (26), one obtains that

$$z_{n+1} = (4f_n + 2\sqrt{3}\,g_n) \tag{31}$$

Thus,(29) ,(30) and (31) represent the integer solutions to (1).

Example 6:

Considering the value of k to be 7 in (26),it gives the negative pell equation

$$y^2 = 48x^2 - 12 \tag{32}$$

In this case, the corresponding integer solutions to (32) are given by

$$y_{n+1} = (3f_n + 2\sqrt{3}g_n) \tag{33}$$

$$x_{n+1} = \frac{1}{4}(2f_n + \sqrt{3}g_n) \tag{34}$$

where

$$f_n = (7+\sqrt{48})^{n+1} + (7-\sqrt{48})^{n+1} \;, g_n = (7+\sqrt{48})^{n+1} - (7-\sqrt{48})^{n+1}$$

Using (34) in (26), one obtains that

$$z_{n+1} = \frac{7}{4}(2f_n + \sqrt{3}g_n) \tag{35}$$

Thus,(33) ,(34) and (35) represent the integer solutions to (1).

Illustration 6:

The substitution of the linear transformations

$$y = kx, k \neq 1 \tag{36}$$

in (1) leads to the positive pell equation

$$z^2 = (k^2+1)\ x^2 + 12 \tag{37}$$

Example :7

Considering the value of k to be 6 in (36),it gives the positive pell equation

$$z^2 = 37x^2 + 12 \tag{38}$$

After some algebra ,the corresponding integer solutions to (38) are given by

$$x_{n+1} = \frac{1}{2}(f_n + \frac{7}{\sqrt{37}} g_n) \quad (39)$$

$$z_{n+1} = \frac{1}{2}(7f_n + \sqrt{37} g_n) \quad (40)$$

$$f_n = (73+12\sqrt{37})^{n+1} + (73-12\sqrt{37})^{n+1}, g_n = (73+12\sqrt{37})^{n+1} - (73-12\sqrt{37})^{n+1}$$

In view of (36), we have

$$x_{n+1} = 3(f_n + \frac{7}{\sqrt{37}} g_n) \quad (41)$$

Thus,(39) ,(40) and (41) represent the integer solutions to (1).

Illustration 7:

The substitution of the linear transformations

$$x = y + h, h \neq 0 \quad (42)$$

in (1) leads to the negative pell equation

$$(2y+h)^2 = 2z^2 - (h^2 + 24) \quad (43)$$

for which the integer solutions exist when h takes particular values.

Example 8:

Considering the value of h to be 1 in (42),it gives the negative pell equation

$$(2y+1)^2 = 2z^2 - 25 \quad (44)$$

In this case, the corresponding integer solutions to (44) are given by

$$y_{n+1} = \frac{1}{4}(5f_n + 5\sqrt{2} g_n - 2) \quad (45)$$

$$z_{n+1} = \frac{1}{2}(5f_n + \frac{5}{\sqrt{2}} g_n) \quad (46)$$

where

$$f_n = (3+2\sqrt{2})^{n+1} + (3-2\sqrt{2})^{n+1} \ , g_n = (3+2\sqrt{2})^{n+1} - (3-2\sqrt{2})^{n+1}$$

Using (45) in (42), one obtains that

$$x_{n+1} = \frac{1}{4}(5f_n + 5\sqrt{2}\, g_n + 2) \tag{47}$$

Thus,(45),(46) and (47) give the integer solutions to (1).

Example 9:

Considering the value of h to be 2 in (42),it gives the negative pell equation

$$(2y+2)^2 = 2\ z^2 - 28 \tag{48}$$

In this case, the corresponding integer solutions to (48) are given by

$$y_{n+1} = \frac{1}{2}(f_n + 2\sqrt{2}\, g_n - 2) \tag{49}$$

$$z_{n+1} = (2f_n + \frac{1}{\sqrt{2}} g_n) \tag{50}$$

where

$$f_n = (3+2\sqrt{2})^{n+1} + (3-2\sqrt{2})^{n+1} \ , g_n = (3+2\sqrt{2})^{n+1} - (3-2\sqrt{2})^{n+1}$$

Using (49) in (42), one obtains that

$$x_{n+1} = \frac{1}{2}(f_n + 2\sqrt{2}\, g_n + 2) \tag{51}$$

Thus,(49),(50) and (51) give the integer solutions to (1).

Conclusion :

In this paper ,an attempt has been made to obtain different sets of non-zero distinct integer solutions to the ternary quadratic diophantine equations $x^2 + y^2 = z^2 - 12$. As diophantine equations are rich in variety ,the readers of this paper may search for choices of the integer solutions to the other forms of ternary quadratic diophantine equations.

References :

1) L.E. Dickson, History of theory of Numbers, Vol. 2, Chelsea publishing Company, Newyork, 1952.
2) L.J. Mordel, Diophantine Equations, Academic press, Newyork, 1969.
3) M.A.Gopalan and V.Pandichelvi, On the ternary quadratic equation $x^2 + y^2 = z^2 + 1$, Impact J.Sci.Tech: vol 2(2), 55-58,2008.
4) M.A. Gopalan and P. Shanmuganandham, Integer solutions of ternary quadratic equations $x^2 + y^2 = z^2 - 4$, Impact J.Sci.Tech: vol2(2), 59-63, 2008
5) M.A.Gopalan and P.Shanmuganandham , Integer Solutions of Ternary Quadratic equation $x^2 + y^2 = z^2 + 4$, Impact J.Sci.Tech : vol2(3) ,139-141,2008.
6) M.A.Gopalan and J.Kaliga Rani, On the ternary quadratic equation $x^2 + y^2 = z^2 + 8$ Impact J.Sci.Tech: vol 5, No.1, 39-43,2011.
7) S.Vidhyalakshmi , M.A.Gopalan , Observations On The Paper Entitled “Integer solution of Ternary Quadratic Equation $x^2 + y^2 = z^2 - 4$ “ International Journal of Current Science , Vol 12 , Issue 1, 401-406, January 2022.
8) S.Vidhyalakshmi , M.A.Gopalan , Observations on the integer solutions of Ternary Quadratic Equation $x^2 + y^2 = z^2 - 8$,IJMCR , Vol 10 , Issue 5, 2690-2692,May 2022.
9) A.Vijayasankar,Sharadha Kumar,M.A.Gopalan ,Observations on the Integral Solutions of Ternary Quadratic Equation $x^2 + y^2 = z^2 + 12$,IJRPR ,Vol 3 ,Issue 2 , 808-814 ,2022
10) S.Vidhyalakshmi , M.A.Gopalan , On finding Integral Solutions of Ternary Quadratic Equation $x^2 + y^2 = z^2 - 2k^2$,IJRRMCSIT , Vol 9 , Issue 1, 16-19, 2022
11) J.Shanthi ,M.A.Gopalan P.Dhanassree ,Observations On The Integral Solutions Of The Ternary Quadratic Equation $x^2 + y^2 = z^2 + 10$ IRJEdT ,Vol 4 ,Issue 7 ,220-231 ,2022
12) S.Vidhyalakshmi ,M.A.Gopalan ,On Non-homogeneous Ternary Cubic Diophantine Equation $w^2 + 5z^2 - 2wx - 10zx = 6x^3 - 6x^2$,IJRPR ,Vol 3 ,No 8 ,1308-1310 ,2022
13) S.Vidhyalakshmi ,M.A.Gopalan ,On Non-homogeneous Ternary Cubic Diophantine Equation $w^2 + 2z^2 - 2wx - 4zx = 9x^3 - 3x^2$,JRMMA ,Vol 1 ,Issue 4 ,01220104004-1-01220104004-2, ,2022