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THE SYLOW THEOREMS

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Abstract:

I am introducing in this paper the concept of group actions, along with the associated notions of orbits and the Sylow theorem, I build up to the three Sylow theorems, and subsequently give some of their examples.

Index Terms: p-subgroup, normal subgroup, proper subgroup, Finite Group, cyclic groups and group action.

I. INTRODUCTION

In a group theory Lagrange's theorem is one of the important theorems in finite groups, which states that the order of any subgroup of a group divides the order of the group. But the converse of this theorem is not true, that for any number dividing the order of a group, there exists a subgroup of that order. The Sylow theorems do gives us partial converse to Lagrange's theorem, by asserting the existence of certain subgroups (called Sylow p-subgroups) of any group with a given order, and gives some information about their properties. In this paper, we wish to build up to the knowledge of the Sylow theorems. A group G is said to be a p-group (p is prime number), if the order of every element of G is some power of p.

SYLOW THEOEMS :

First Sylow theorem States :

If p^m divides the order of a finite Group G (p being a prime number), then G has a subgroup of order p^m. We know that the group is simple if it has no proper normal subgroups. It is applied to determine whether a given finite group is simple or not.

Theorem:

If p is a prime number and p^m divides o(G) , then G has subgroup of order p^m

Note:

A particular case of the above theorem is the following: If p is a prime number such that p^m/o(G) and p^{m+1}†o(G) , then G has asubgroup of order pm.. Sylow p- subgroup (Definition) : If G is the finite group and p is a prime number, then a subgroup of G of order pm , then a subgroup of G of order p^m, where pm/o(G) but pm+1†o(G) is called sylow p- subgroup or p-sylow subgroup of Gor in short p-SSG of G.

Note:

The above definition implies that all sylowp- subgroups of G are of the same order i.e. p^m. Since any p-subgroup of G is oforder, a power of p, it follows that, No subgroup of G can contain properly a sylow p-subgroup of G.

Note:

By the above statement, it follows that A finite group G has a p- sylow subgroup for each prime p which divides o(G).

Remark:

The condition p^m/o(G) and p^{m+1}†o(G) imply that m is the highest power of p such that p^n/o(G) . Equivalently the above conditions enable us to write o(G) as follows :

$$O(G) = p^m.n, \text{ where } p \dagger n$$

The first sylow theorem can be restates as:

If $o(G) = p^m.n$, then G has a sylow p-subgroup of order pm. This form is helpful in finding sylow p-subgroup of a finite group.

Example:

i) ,

If o(G) = 48 = 16*3=24*3, Where 2†3; then G has a 2-SSG of order 24= 16. Also G has a 3- SSG of order 3.

ii),

If o(G) = 56 = 23* 7, then G has a 2-SSG of order 23= 8 and a 7-SSG of order 7.

Note :

From the above we note that , The first sylow theorem tells us as to which type of p-sylow subgroup a given finite group possesses.

Lemma;

(Double Coset Decomposition): If A and B are two subgroupof a group G, then G = $\bigcup_{x \in G} AxB$

Lemma:

If A and B are finite subgroups of a group G, then

$$G = \bigcup_{x \in G} AxB \qquad \text{Then } o(AxB) = \frac{o(A)o(B)}{o(A \cap xBx^{-1})}, \quad x \in G,$$

Second Sylow Theorem :

Any two sylow p-subgroup of a finite group G is unique if and only if it is normal.

If H is the only p-SSG of G, then H is normal in G.

Lemma:

Let P be a sylow p- subgroup of a finite group G. then the number n_p of Sylow p-subgroup of G is given by $n_p = 1+kp$ and n_p divides o(G).

Note :

The first Sylow theorem tells us as to what type of Sylow p-subgroups, a finite group G can have. The Third Sylow theorem will tell us as to how many sylow p-subgroups, G can have.

Third Sylow Theorem :

Show that the number n_p of Sylow p-subgroup of a finite group G is given by $n_p= 1+ kp$, Where k = 0, 1, 2,… and n_p divides o(G).

Rule:

The number (n_p) of p-SSG of G is given by $n_p=1+ kp$; k = 0, 1, 2,… and n_p divides o(G).

Remark:

The formula for k = 0 is always true, for $n_p = 1$ and1 divides o(G). The problem becomes more interesting if we find some positive integral value of k such that $n_p= 1+kp$ and n_p divides o(G).

First sylow theorem:

If G is finite group such that p^n/o(G) and p^{m+1}†o(G) (p being prime) then G has a subgroup H of order p^n. H is called a Sylow p- subgroup of G.

Second Sylow theorem:

Any two Sylow p-Subgroups of a finite group G are conjugate in G i.e. If P and Q are two sylow p-subgroup of G then Q = xpx-1 for some.

Third Sylow theoem:

The number np of sylow p-subgroup of a finite group G is given by np=1+ kp ; k = 0, 1, 2,… and np divideso(G).

Example;

Prove that the group of order 28 has a group of order 7. And hence prove that a group of order 28 is not simple

.

Solution:

We have o(G) = 28 = 7*22By First sylow theorem , 7-SSG is given by n7= 1 + 7k, k = 0, 1, 2,… and n7 divides o(G) =28.

For k=0, n7= 1 and n7
divides order of G =28 .For
k= 1,n7 =8 but 8 does not
divide o(G)= 28.
For k= 2,n7 =15 but 15 does not divide o(G)= 28. And so on.

Therefore, n7= 1 i.e. there exist exactly one7-SSG say H, where, o(H) = 7. Hence H is a normal Subgroup of order 7. Recall thatgroup G is simple if it has no proper normal subgroup.

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