International Journal of Research Publication and Reviews

Journal homepage: www.ijrpr.com ISSN 2582-7421

STATISTICAL METHODS FOR INSURANCE MODELLING UNDER CLAIM AND SURPLUS INSURANCE PROCESSES FOR RUIN PROBABILITIES

G. Jagath Prasad Sreedhar [1], V. Nirmala [2] and K. K. Suresh [3]

[1] Ph.D., Research Scholar, Department of Statistics, Bharathiar University, Coimbatore - 641 046, Tamil Nadu

[2] Corresponding Author, Assistant Professor and Scientist - A, Department of Statistics, Sri Krishna Arts and Science College, Coimbatore, Tamil Nadu. E-mail: nirmalastat5@gmail.com , Contact No.: 97870 99896.

[3] Research Supervisor, Professor and Head (Retd.) Department of Statistics, Bharathiar University, Coimbatore – 641 046, Tamil Nadu.

Abstract:

In this paper presents the generalized linear models, many problems in actuarial statistics are Generalized Linear Models (GLM). In- stead of assuming a normally distributed error term, other types of randomness are allowed as well, such as Poisson, gamma and binomial. Also, the expected values ofthe dependent variables need not be linear in the regressors. They may also be some function of a linear form of the covariates, for example the logarithm leading to the multiplicative models that are appropriate in many insurance situations. This way, one can for example tackle the problem of estimating the reserve to bekept for IBNR claims, see below. But one can also easily estimate the premiums tobe charged for drivers from region i in bonus class j with car weight w.

Key words: Ruin theory, Insurance claim, Renewal, Dependent risk model, surplus processes.

Introduction

The insurance industry exists because people are willing to pay a price for being insured. There is an economic theory that explains why insured are willing to pay a premium larger than the net premium, that is, the mathematical expectation of the insured loss. This theory postulates that a decision maker, generally without being aware of it, attaches a value u(w) to his wealth w instead of just w, where u(·) is called his utility function

Insurance companies maintain solvency via careful design of premium rates. The premium rates are primarily based on the claims history and carefully adjusted to evolving factors, such as the number of customers and/or the returns from the investments in the financial market. Collective risk models, introduced by Lundberg and Cramér, describe the evolution of the surplus of an insurance business when considering constant premium rates, for the simplicity of arguments. This model, a compound Poisson process with drift, is referred to in the actuarial mathematics literature as the Cramér-Lundberg model. In practical situations, risk models with surplus-dependent premiums capture the dynamics of the surplus of an insurance company better. The Reference Lin and Pavlova (2006) advised for a lower premium for higher surplus levels to improve competitiveness, whereas a higher premium is needed for lower surplus levels to reduce the probability of ruin.

Among surplus-dependent premiums, risk models with *risky* investments have been widely analyzed (see e.g., Albrecher et al. 2012; Frolova et al. 2002; Paulsen 1993; Paulsen and Gjessing 1997). See Paulsen (1998) and Paulsen (2008) for surveys on the topic. The special case of risk models with *linearly* dependent premiums can be interpreted as models with *riskless* investments, since the volatility of return on investments, or the proportion of the capital invested in the risky asset is zero.

Under this scenario, exact expressions of the ruin probability are derived for compound Poisson risk models with interest on surplus and exponential-type upper bounds for renewal risk models with interest (see Cai and Dickson 2002, 2003). The Reference Cheung and Landriault (2012) investigated risk models with surplus-dependent premiums with dividend strategies and interest earning as a special case. Nirmala and Suresh (2018) proposed Designing of MATLAB Program for Various Fuzzy Quality Regions in CSP-MLP-T-3 Sampling plan. The Reference Czarna et al. (2019) discussed the ruin probabilities with the scale function from the theory of the Lévy process for risk models when the claim arrival process is a spectrally negative Lévy process and the premium rate function is non-decreasing and locally Lipschitz-continuous.

During the last two decades, dependence has been increasingly playing an important role in the world of risk, especially in the insurance field. The classical Cramer-Lundberg model gives the surplus process of an insurance portfolio and is based on the assumption of independence among claim sizes and between claim size and claim interval time. Anderson [6] generalized the compound Poisson model by assuming that inter claim times of random variables are only independently identically distributed. While independence can be defined in only one way, dependence can be formulated in an unlimited number of ways. Traditionally, risk theory assumes independence between the different variables of interest. However, in many situations insured risks usually behave in a similar manner. For example, consider a group health insurance contract issued to a company for a section of its employees working in a mine and occurring of a single event like explosion influences the risks of the entire portfolio. So when modelling natural events, this assumption is too restrictive and far away from real scenario. After that, many researchers relaxed the assumption of independence between the claim amounts and the inter claim times, and studied the impact of dependence on the survival probabilities.

The Model

The general surplus or risk process of the insurance portfolio is defined as follows: Let $u > 0$ represents the initial capital, and c denotes the premium which is assumed to be a positive constant, then the surplus process of an insurance company, $U(t)$ can be expressed as

$$U(t) = u + ct - X(t)$$

Where the aggregated claim process $X(t)$ can be expressed as $X(t) = \sum_{i=1}^{N(t)} X_i$, with X_i is the i^{th} claim size and $N(t)$ is a Poisson process with intensity λ counting the number of claims up to time t [8,9,10]. Let $\{T_i, i = 1,2,..\}$ be a sequence of claim inter occurrence times with threshold $\{A_i\}$, where $\{A_i, i = 1,2,..\}$ be a sequence of i.i.d non negative random variables. Assume that claim sizes are determined in such a way that X_i has distribution function $F_1(x)$ with parameter β_1 if $T_i < A_i$, otherwise claim size X_i is distributed as $F_2(x)$ with parameter β_2 . We also assume that the random variables from the set $\{A_i, i = 1,2,..\}$ are jointly independent with $\{T_i, i = 1,2,..\}$ and $\{F_i, i = 1,2\}$.

Since $N(t)$ is a Poisson process with intensity λ , it is obvious that $\{T_i, i = 1,2,..\}$ are exponentially distributed with parameter λ since homogeneous Poisson process is a renewal process with exponentially distributed inter-arrival times. Let us assume that $\{A_i, i = 1,2,..\}$ also distributed exponentially with parameter γ , since T_i have a threshold value A_i *for* $i=1,2,\ldots$. Then a positive expected net profit condition is that

$$\frac{c}{\lambda} > \frac{2P(A>T)}{\beta_1} + \frac{2P(A\le T)}{\beta_2}$$

where A represents the identical distribution for all $A_i, i=1,2,..$ and $P(A>0) = P(T>0) = 1$. Let $\Phi(u)$ denotes the probability of survival with initial capital u given that the first claim inter occurrence time distributed exponentially with rate λ . The time of ruin is a stopping time given by $T = \inf\{t>0: U(t)<0\}$ and the probability of ultimate ruin be denoted by $\Psi(u)$ where $\Psi(u)$ is defined as

$$\Psi(u) = P\{Ruin / U(0) = u\} = P\{T < \infty\}$$

Recall the fact that the ruin probability can be explicitly calculated from $1-\Phi(u)$, and hence the survival probability can also be expressed in terms of the general surplus as

$$\Phi(u) = P\{U(t) \ge 0,\ \forall\ t \ge 0\} = P(T=\infty)$$

The availability of analytical solution for the survival probability helps to identify the dependency structure within the model, indeed corresponding independent structure can easily be deduced. Hence we obtain an analytical procedure to derive explicit expression.

for the defined model with Erlang(2) distribution having mean $\frac{2}{\beta_1}, i=1,2$ for the claim size X_i , $i=1,2$ under threshold condition.

Theorem 1: The survival probability on the first jump time of the surplus process can be expressed in terms of an integral equation as

$$\Phi(u) = \frac{\beta_1^2\lambda}{c}\int_u^\infty \exp\left\{-\frac{\lambda(s-u)}{c} - \frac{\gamma(s-u)}{c}\right\}\int_0^s \Phi(s-x)\,x\,e^{-\beta_1 x}\,dx\,ds$$

$$+\ \frac{\beta_2^2\lambda}{c}\int_u^\infty \exp\left\{-\frac{\lambda(s-u)}{c}\right\} - \exp\left\{-\frac{\lambda(s-u)}{c} - \frac{\gamma(s-u)}{c}\right\}\int_0^s \Phi(s-x)\,x\,e^{-\beta_2 x}\,dx\,ds \quad (1)$$

where $\Phi(u)$ represents the ultimate survival probability and $\Re(s) \ge 0$, the premium rate c can be expressed as $c = \frac{s-u}{t}$, u be the initial capital at time t .

Note: The integral equation in (1) leads to the derivation of explicit expression for the survival probability using differential calculus method. The equation (1) yields dependency structure in the claim size and threshold condition provides the distribution of claim sizes has to be distributed as Erlang with parameters β_1 and β_2 respectively.

Proof: For having positive expected profit, the net profit condition should satisfy

$$\frac{c}{\lambda} > E(F_1 \setminus A > T).P(A > T) + E(F_2 \setminus A \leq T).P(A \leq T)$$

where $F_i, i = 1,2$ are the threshold distribution corresponds to A when $T < A$ and $T \geq A$ respectively. Assume that A_i follows Exponential (γ), F_1 and F_2 are Erlang distributions with parameter β_1 and β_2 respectively. By considering the first jump time of the surplus process at time t, we get the following integral equation.

$$\Phi(u) = \lambda \int_0^{\infty} e^{-\lambda t} \int_0^{u+ct} P(A > t)\, \Phi(u + ct - x)\, F_1(dx)\, dt$$

$$+ \lambda \int_0^{\infty} e^{-\lambda t} \int_0^{u+ct} P(A \leq t)\, \Phi(u + ct - x)\, F_2(dx)\, dt$$

since the initial surplus for the first jump can be represented up to $u + ct$. Assume that $s = u + ct$ where $\Re(s) \geq 0$,

$$\Phi(u) = \frac{\lambda}{c} \int_u^{\infty} e^{-\lambda t} \int_0^{s} P(A > \frac{s-u}{t})\, f_1(x)\, \Phi(s - x)\, dx\, ds$$

$$+ \frac{\lambda}{c} \int_u^{\infty} e^{-\lambda t} \int_0^{s} \{P(A \leq \frac{s-u}{t})\, f_2(x)\, \Phi(s - x)\, dx\, ds$$

where $f(.)$ denotes probability density function (pdf) corresponding to $F(.)$. For Erlang distribution with $n = 2$ and parameter β_1 , the pdf is given as ${\beta_1}^2 x e^{-\beta_1 x}$ for $x \geq 0$ and β_1 being integer. Substitute the pdf in equation (2) yields the survival probability as in equation (1) . Hence the proof.

3. Explicit expression for the survival probability, $\Phi(u)$

In this section, we derive the explicit expression for the survival probability of an insurance portfolio where the claim inter-arrival time depends on the next claim size that follows an Erlang distribution, using the differential calculus method and by applying Laplace transform properties [25]. For $\Re(s) \ge 0$, the Laplace transform of a function $\phi(.)$ is defined as $\phi(x) = \int_0^\infty e^{-sx} \phi(x)\, dx$. Throughout the paper $h'(x)$ represents the first derivative of the function $h(.)$ with respect to x and $h''(x)$ represents the second derivative of the function $h(.)$ with respect to x unless specified. Re-arranging the survival probability $\Phi(u)$ in equation (2) , we get,

$$\frac{c}{\lambda}\Phi(u) = \beta_1^2 \int_u^\infty \exp\left\{-\frac{\lambda(s-u)}{c} - \frac{\gamma(s-u)}{c}\right\} \int_0^s \Phi(s-x)\, x\, e^{-\beta_1 x}\, dx\, ds$$

$$+ \beta_2^2 \int_u^\infty \exp\left\{-\frac{\lambda(s-u)}{c}\right\} - \exp\left\{-\frac{\lambda(s-u)}{c} - \frac{\gamma(s-u)}{c}\right\} \int_0^s \Phi(s-x)\, x\, e^{-\beta_2 x}\, dx\, ds \quad (3)$$

Now differentiate both sides of equation (3) with respect to u and re-arranging terms, we get the following integro - differential equation:

$$\frac{c}{\lambda}\Phi'(u) = -\beta_1^2 \int_0^u \Phi(s-x)\, x\, e^{-\beta_1 x}\, dx\, ds$$

$$+ \frac{\beta_1^2(\lambda+\gamma)}{c} \int_u^\infty \exp\left\{-\frac{(\lambda+\gamma)(s-u)}{c}\right\} \int_0^s \Phi(s-x)\, x\, e^{-\beta_1 x}\, dx\, ds$$

$$+ \frac{\beta_2^2 \lambda}{c} \int_u^\infty \exp\left\{-\frac{\lambda(s-u)}{c}\right\} \int_0^s \Phi(s-x)\, x\, e^{-\beta_2 x}\, dx\, ds$$

$$- \frac{\beta_2^2(\lambda+\gamma)}{c} \int_u^\infty \exp\left\{-\frac{(\lambda+\gamma)(s-u)}{c}\right\} \int_0^s \Phi(s-x)\, x\, e^{-\beta_2 x}\, dx\, ds \quad (4)$$

Substituting $u-x$ as y in equation (4) and consider $\Phi(u)$ as in equation (3). Simplifying and re-arranging terms, we got the following equation:

$$\frac{c}{\lambda}\Phi'(u) = \frac{(\lambda+\gamma)}{\lambda}\Phi(u) - \beta_1^2 \int_0^u \Phi(u-x)\, x\, e^{-\beta_1 x}\, dx\, ds$$

Let us consider that $\widetilde{\Phi}(s)$ is a ratio of two polynomials such that $\widetilde{\Phi}(s) = \dfrac{h(s)}{f(s) - g(s)}$ where the numerator $h(s) = \frac{c}{\lambda}\Phi'(0) + \frac{c}{\lambda} s\,\Phi(0) - \left(2 + \frac{\gamma}{\lambda}\right)\Phi(0)$ and the denominator $f(s) = \frac{c}{\lambda}s^2 - \left(2 + \frac{\gamma}{\lambda}\right)s + \left(\frac{\lambda+\gamma}{c}\right)$ and

$g(s)=\left(\frac{\lambda}{c}+\beta_1\right)\frac{\beta_1^2}{(s+\beta_1)^2}+\frac{\gamma\beta_2^2}{c(s+\beta_2)^2}-\frac{\beta_1^2}{(s+\beta_1)}$. Clearly, $\widetilde{\Phi}(s)$ is a proper rational function of s where the degree of numerator function $h(s)$ is less than that of the denominator $f(s)-g(s)$.

The survival probability, a decisive factor of insurance companies is simulated using R software [30]. We generate values of $\Phi(u)$ for different sequences of initial capital u . The sequence of u are 0,1,...,10 with an increment of 1, 0,2,...,20 with an increment of 2 and 0,5,…,50 with an increment 5, to understand the spanning of the survival probability for different values of the surplus in the economy. The average and variance within entire span are also calculated. The following table 1 illustrate different values of initial capital along with ultimate survival probability for example 1, where we can see that the variance of the surplus process are equal to 0.04 for all span of u . The application of probability to an insurance portfolio under a dependent setting for the claim sizes and claim interval times in a renewal risk model, where the time between two claim sizes determines the next claim size.

An explicit solution for the probability of survival of the insurance companies is derived for claim inter interval times and Erlang(2) distributed claim sizes, using Laplace-Stieltjes transform. Its deals with insurance portfolio's with dependent risks. During the last two decades, dependence has been increasingly playing an important role in the world of risk, especially in the insurance field. The classical Cramer-Lundberg model gives the surplus process of an insurance portfolio and is based on the assumption of independence among claim sizes.

Generalized Linear Models have three characteristics:

Claim size and claim interval time. Anderson [6] generalized the compound Poisson model by assuming that inter claim times of random variables are only independently identically distributed. While independence can be defined in only one way, dependence can be formulated in an unlimited number of ways. Traditionally, risk theory assumes independence between the different variables of interest. However, in many situations insured risks usually behave in a similar manner. The stochastic component of the model states that the observations are independent random variables Y_i, i = 1,…, n with a density in the exponential dispersion family.

Illustrate the ideas behind GLMs using I J contingency tables. We have a table of observed insurance losses Y_{ij}, i = 1,…, I, j = 1,…, J, classified by two rating factors into I and J risk classes. Hence, we have I J inde- pendent observations (but some cells may be empty) indexed by i and j instead of nobservations indexed by i as before. Generalization to more than two dimensions is straightforward. The collateral data with each observation consist of the row number i and the column number j in the table. The numbers in each cell represent averages over all w_{ij} observations in that cell (natural weights). With these factors, we try to construct a model for the expected values of the observations. Many situations are covered by this example. For example, the column number may indicate a certain region/usage combination such as the row number may be a weight class for a car or a step in the bonus-malus scale. The observations might then be the observed average number of accidents for all drivers with the characteristics i and j. Other examples, see also the next chapter, arise if i is the year that a certain policy was written, j is the development year, and the observations denote the total amount paid in year i + j 1 regarding claims pertaining to policies of the year i. The calendar year i + j 1 can be used as a third collateral variable. Successive substitution can be implemented in R quite easily.

Conclusion

The probability of ruin enables one to compare portfolios, but cannot attach any absolute meaning to the probability of ruin, as it does not actually represent the probability that the insurer will go bankrupt in the near future. First of all, it might take centuries for ruin to actually happen. Second, obvious interventions in the process such as paying out dividends or raising the premium for risks with an unfavourable claims performance are ruled out in the definition of the probability of ruin. Furthermore, the effects of inflation and return on capital are supposed to cancel each other out exactly.

Acknowledgment

The Authors like to acknowledge, Bharathiar University for providing necessary facilities for pursuing Ph.D. research programme Statistics.

Financial support and sponsorship

Nil.

Declaration of competing interest

All authors declare that they have no conflict of interests

References

- Abramowitz, Milton, and Irene A. Stegun. 1965. Handbook of Mathematical Functions: With Formulas, Graphs, and Mathematical Tables. New York: Dover Publications.

- Albrecher, Hansjoerg, Corina Constantinescu, and Enrique Thomann. 2012. Asymptotic results for renewal risk models with risky investments. Stochastic Processes and their Applications 122: 3767–89.
- Albrecher, Hansjörg, Corina Constantinescu, Gottlieb Pirsic, Georg Regensburger, and Markus Rosenkranz. 2010. An algebraic operator approach to the analysis of Gerber–Shiu functions. Insurance: Mathematics and Economics 46: 42–51.
- Albrecher, Hansjörg, Corina Constantinescu, Zbigniew Palmowski, Georg Regensburger, and Markus Rosenkranz. 2013. Exact and asymptotic results for insurance risk models with surplus-dependent premiums. SIAM Journal on Applied Mathematics 73: 47–66.
- Asmussen, Søren, and Hansjörg Albrecher. 2010. Ruin Probabilities. Singapore: World Scientific Singapore, vol. 14.
- Bergel, Agnieszka I., and Alfredo D. Egidio dos Reis. 2015. Further developments in the Erlang (n) risk process. *Scandinavian Actuarial Journal* 2015: 32–48.
- Cai, Jun, and David C. M. Dickson. 2002. On the expected discounted penalty function at ruin of a surplus process with interest. *Insurance: Mathematics and Economics* 30: 389–404.
- Cai, Jun, and David C. M. Dickson. 2003. Upper bounds for ultimate ruin probabilities in the Sparre Andersen model with interest. *Insurance: Mathematics and Economics* 32: 61–71.
- Cheung, Eric C. K., and David Landriault. 2012. On a risk model with surplus-dependent premium and tax rates. *Methodology and Computing in Applied Probability* 14: 233–51.
- H.U. Gerber, H.L. Yang, Absolute ruin probabilities in a jump diffusion risk model with investment, N. Am. Actuar. J. 11 (2007) 159–169.
- I. Coulibaly, C. Lefèvre, On a simple quasi-Monte Carlo approach for classical ultimate ruin probabilities, Insur. Math. Econ. 42 (2008) 935–942
- K. Shakenov, Solution of Equation for Ruin Probability of Company for Some Risk Model by Monte Carlo Methods, Springer, 2016.
- Nirmala.V (2020). Evaluations of Continuous Sampling Plan indexed through Maximum Allowable Average Outgoing Quality, International Journal of Advanced Science and Technology. Vol. 29 No. 2, pp 3624-3633.
- Nirmala.V and Suresh. K. K (2016). Designing of continuous sampling plan (CSP-T) through quality decision regions, International Journal of Pure and Applied Mathematics. Vol. 106 No. 7, pp 15-20.

- Nirmala.V and Suresh. K. K (2017). Construction and selection of continuous sampling plan of type (CSP-T) indexed through maximum allowable average outgoing quality. Journal of Statistics & Management System. Vol. 20 No. 3, pp 441-457.
- Nirmala.V and Suresh. K. K (2017). Designing of Matlab Program for Continuous Sampling Plan - F Indexed Through Maximum Allowable Average Outgoing Quality, International Journal of Pure and Applied Mathematics. Vol. 117 No. 13, pp 241-250.
- Nirmala.V and Suresh. K. K (2018). A case study on early detection, prediction and prevention of heart disease in a multispecialty hospital by applying six sigma methodologies. International Journal of Scientific Research in Mathematical and Statistical Sciences. 05(04): 01-08.
- Nirmala.V and Suresh. K.K (2015): "International Journal of Science and Research", Titled: "Comparison of Certain Type of Continuous sampling Plans (CSPs) and its operating procedure-A Review", Vol.4, Issue 3, pp.455-459, Impact factor: 4.43.
- Nirmala.V and Suresh. K.K (2017): "International Journal of Statistics and Systems (IJSS)", Titled: "Designing of Continuous Sampling Plan of type (CSP-3) indexed through Maximum Allowable Average Outgoing Quality with Markov chain approach", ISSN 0973-2675, Vol. 12, Issue 2, pp. 277-283.
- Nirmala.V and Suresh. K.K (2017): "International Journal on Future Revolution in Computer Science and Communication Engineering (IJFRCSCE)", Titled: "Designing of MATLAB Program for Various Fuzzy Quality Regions in CSP-MLP-T-3 Sampling plan", Vol. 3, Issue 7, pp. 065-068.
- Nirmala.V and Suresh. K.K (2017): Far East Journal of Theoretical Statistics (FEJTS)", Titled: "Determination of Maximum Allowable Average Outgoing Quality (MAAOQ) for Continuous Sampling Plan (CSP-1)", Current Index Journal. Vol.52, Issue 1, pp.49-59.
- Nirmala.V and Suresh. K.K (2018): "International Journal of Scientific Research in Mathematical and Statistical Sciences", Titled: "Designing of Modified Tightened 3 Levels Continuous Sampling Plan Indexed through Maximum Allowable Average Outgoing Quality", Vol: 05, Issue: 05, pp. 19 – 24.
- P. Embrechts, H. Schmidli, Ruin estimation for a general insurance risk model, Adv. Appl. Probab. 26 (2) (1994) 404–422.

- R.M.R. Cardosoa, H.R. Waters, Calculation of finite time ruin probabilities for some risk models, Insur. Math. Econ. 37 (2005) 197–215.
- S. Asmussen, H. Albrecher, Ruin Probabilities, World Scientific, 2010.
- S. Asmussen, K. Binswanger, Simulation of ruin probabilities for subexponential claims, ASTIN Bull. 27 (1997) 297–318.