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On The Non-Homogeneous Second Degree Equation

$$y^2 = 15x^2 + 16$$

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ABSTRACT:

The binary quadratic equation represented by the Positive Pellian $y^2 = 15x^2 + 16$ is analyzed for its distinct integer solutions. A few interesting relations among the solutions are given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas, parabolas and special Pythagorean triangle.

KEYWORDS: Binary quadratic, hyperbola, parabola, integral solutions, Pell equation.

2010 Mathematics Subject Classification: 11D09.

INTRODUCTION:

The binary quadratic equation of the form $y^2 = Dx^2 + 1$, where D is a non-square positive integer has been studied by various mathematicians for its non-trivial integral solutions when D takes different integral values [1-4]. For an extensive review of various problems, one may refer [5-21]. In this communication, yet another interesting hyperbola given by $y^2 = 15x^2 + 16$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are presented.

METHOD OF ANALYSIS:

The positive Pell equation representing hyperbola under consideration is

$$y^2 = 15x^2 + 16 \tag{1}$$

Whose smallest positive integer Solution is $x_0 = 4, \quad y_0 = 16$

To obtain the other solutions of (1), consider the Pell equation $y^2 = 15x^2 + 1$ whose solution is given by

$$\tilde{x}_n = \frac{1}{2\sqrt{15}} g_n, \quad \tilde{y}_n = \frac{1}{2} f_n$$

Where

$$f_n = \left(4 + \sqrt{15}\right)^{n+1} + \left(4 - \sqrt{15}\right)^{n+1}$$

$$g_n = \left(4 + \sqrt{15}\right)^{n+1} - \left(4 - \sqrt{15}\right)^{n+1}, n = -1, 0, 1, 2, \ldots\ldots$$

Applying Brahmagupta Lemma between (x_0, y_0) and $(\tilde{x}_0, \tilde{y}_0)$, the other integer solutions of (1) are given by

$$\Rightarrow \sqrt{15} x_{n+1} = 2\sqrt{15} f_n + 8 g_n$$

$$\sqrt{15} y_{n+1} = 8\sqrt{15} f_n + 30 g_n$$

The recurrence relations satisfied by x and y are given by

$$x_{n+3} - 8x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 8y_{n+2} + y_{n+1} = 0$$

A few numerical examples are given in the following Table

Table1: Numerical examples

n	x_n	y_n
0	4	16
1	32	124

2	252	976
3	1984	7684
4	15620	60496

From the above table, we observe some interesting properties among the solutions which are presented below:

1) x_n and y_n values are always even.

2) Relations among the solutions

- $x_{n+1} - 8x_{n+2} + x_{n+3} = 0$
- $4x_{n+1} - x_{n+2} + y_{n+1} = 0$
- $x_{n+1} - 4x_{n+2} + y_{n+2} = 0$
- $4x_{n+1} - 31x_{n+2} + y_{n+3} = 0$
- $8y_{n+1} + 31x_{n+1} - x_{n+3} = 0$
- $2y_{n+2} + x_{n+1} - x_{n+3} = 0$
- $8y_{n+3} + x_{n+1} - 31x_{n+3} = 0$
- $15x_{n+1} + 4y_{n+1} - y_{n+2} = 0$
- $120x_{n+1} + 31y_{n+1} - y_{n+3} = 0$
- $4y_{n+3} - 31y_{n+2} - 15x_{n+1} = 0$
- $y_{n+1} - 4x_{n+3} + 31x_{n+2} = 0$
- $y_{n+2} - x_{n+3} + 4x_{n+2} = 0$
- $y_{n+3} - 4x_{n+3} + x_{n+2} = 0$
- $4y_{n+2} - y_{n+1} - 15x_{n+2} = 0$

- $y_{n+3} - y_{n+1} - 30x_{n+2} = 0$
- $y_{n+3} - 4y_{n+2} - 15x_{n+2} = 0$
- $31y_{n+2} - 4y_{n+1} - 15x_{n+3} = 0$
- $31y_{n+3} - y_{n+1} - 120x_{n+3} = 0$
- $4y_{n+3} - y_{n+2} - 15x_{n+3} = 0$
- $y_{n+1} - 8y_{n+2} + y_{n+3} = 0$

3) Each of the following expressions represents a nasty number

- $12x_{2n+3} - 93x_{2n+2} + 12$
- $\frac{1}{2}\left(3x_{2n+4} - 183x_{2n+2} + 24\right)$
- $12y_{2n+2} - 45x_{2n+2} + 12$
- $3y_{2n+3} - 90x_{2n+2} + 12$
- $\frac{1}{62}\left(24y_{2n+4} - 5670x_{2n+2} + 744\right)$
- $93x_{2n+4} - 732x_{2n+3} + 12$
- $\frac{1}{4}\left(93y_{2n+2} - 45x_{2n+3} + 48\right)$
- $93y_{2n+3} - 360x_{2n+3} + 12$
- $\frac{1}{4}\left(93y_{2n+4} - 2835x_{2n+3} + 48\right)$
- $\frac{1}{31}\left(732y_{2n+2} - 45x_{2n+4} + 372\right)$
- $183y_{2n+3} - 90x_{2n+4} + 12$
- $732y_{2n+4} - 2835x_{2n+4} + 12$

- $24y_{2n+2} - 3y_{2n+3} + 12$
- $\frac{1}{8}(189y_{2n+2} - 3y_{2n+4} + 96)$
- $189y_{2n+3} - 24y_{2n+4} + 12$

4) Each of the following expressions represents a cubical integer

- $\frac{1}{2}(4x_{3n+4} - 31x_{3n+3} + 12x_{n+2} - 93x_{n+1})$
- $\frac{1}{4}(x_{3n+5} - 61x_{3n+3} + 3x_{n+3} - 183x_{n+1})$
- $\frac{1}{2}(4y_{3n+3} - 15x_{3n+3} + 12y_{n+1} - 45x_{n+1})$
- $\frac{1}{2}(y_{3n+4} - 30x_{3n+3} + 3y_{n+2} - 90x_{n+1})$
- $\frac{1}{124}(8y_{3n+5} - 1890x_{3n+3} + 24y_{n+3} - 5670x_{n+1})$
- $\frac{1}{2}(31x_{3n+5} - 244x_{3n+4} + 93x_{n+3} - 732x_{n+2})$
- $\frac{1}{8}(31y_{3n+3} - 15x_{3n+4} + 93y_{n+1} - 45x_{n+2})$
- $\frac{1}{2}(31y_{3n+4} - 120x_{3n+4} + 93y_{n+2} - 360x_{n+2})$
- $\frac{1}{8}(31y_{3n+5} - 945x_{3n+4} + 93y_{n+3} - 2835x_{n+2})$
- $\frac{1}{62}(244y_{3n+3} - 15x_{3n+5} + 732y_{n+1} - 45x_{n+3})$
- $\frac{1}{2}(61y_{3n+4} - 30x_{3n+5} + 183y_{n+2} - 90x_{n+3})$
- $\frac{1}{2}(244y_{3n+5} - 945x_{3n+5} + 732y_{n+3} - 2835x_{n+3})$
- $\frac{1}{2}(8y_{3n+3} - y_{3n+4} + 24y_{n+1} - 3y_{n+2})$

- $\frac{1}{16}\left(63y_{3n+3}-y_{3n+5}+189y_{n+1}-3y_{n+3}\right)$
- $\frac{1}{2}\left(63y_{3n+4}-8y_{3n+5}+189y_{n+2}-24y_{n+3}\right)$

5) Each of the following expressions represents a bi-quadratic integer

- $\frac{1}{2}\left(4x_{4n+4}-31x_{4n+3}+16x_{2n+2}-124x_{2n+3}+12\right)$
- $\frac{1}{4}\left(x_{4n+6}-61x_{4n+4}+4x_{2n+4}-244x_{2n+2}+24\right)$
- $\frac{1}{2}\left(4y_{4n+4}-15x_{4n+4}+16y_{2n+2}-60x_{2n+2}+12\right)$
- $\frac{1}{2}\left(y_{4n+5}-30x_{4n+4}+4y_{2n+3}-120x_{2n+2}+12\right)$
- $\frac{1}{124}\left(8y_{4n+6}-1890x_{4n+4}+32y_{2n+4}-7560x_{2n+2}+744\right)$
- $\frac{1}{2}\left(31x_{4n+6}-244x_{4n+5}+124y_{2n+4}-976x_{2n+3}+12\right)$
- $\frac{1}{8}\left(31y_{4n+4}-15x_{4n+5}+124y_{2n+2}-60x_{2n+3}+12\right)$
- $\frac{1}{2}\left(31y_{4n+5}-120x_{4n+5}+124y_{2n+3}-480x_{2n+3}+12\right)$
- $\frac{1}{8}\left(31y_{4n+6}-945x_{4n+5}+124y_{2n+4}-3780x_{2n+3}+48\right)$
- $\frac{1}{62}\left(244y_{4n+4}-15x_{4n+6}+976y_{2n+2}-60x_{2n+4}+372\right)$
- $\frac{1}{2}\left(61y_{4n+5}-30x_{4n+6}+244y_{2n+3}-120x_{2n+4}+12\right)$
- $\frac{1}{2}\left(244y_{4n+6}-945x_{4n+6}+976y_{2n+4}-3780x_{2n+4}+12\right)$
- $\frac{1}{2}\left(8y_{4n+4}-y_{4n+5}+32y_{2n+2}-4y_{2n+3}+12\right)$
- $\frac{1}{16}\left(63y_{4n+4}-y_{4n+6}+252y_{2n+2}-4y_{2n+4}+96\right)$

* $\frac{1}{2}\left(63y_{4n+5}-8y_{4n+6}+252y_{2n+3}-32y_{2n+4}+12\right)$

6) Each of the following expressions represents a quintic integer

* $\frac{1}{2}\left[4x_{5n+6}-31x_{5n+5}+20x_{3n+4}-155x_{3n+3}+40x_{n+2}-310x_{n+1}\right]$
* $\frac{1}{4}\left[x_{5n+7}-61x_{5n+5}+5x_{3n+5}-305x_{3n+3}+10x_{n+3}-610x_{n+1}\right]$
* $\frac{1}{2}\left[4y_{5n+5}-15x_{5n+5}+20y_{3n+3}-75x_{3n+3}+40y_{n+1}-150x_{n+1}\right]$
* $\frac{1}{2}\left[y_{5n+6}-30x_{5n+5}+5y_{3n+4}-150x_{3n+3}+10y_{n+2}-300x_{n+1}\right]$
* $\frac{1}{124}\left[8y_{5n+7}-1890x_{5n+5}+40y_{3n+5}-9450x_{3n+3}+80y_{n+3}-18900x_{n+1}\right]$
* $\frac{1}{2}\left[31x_{5n+7}-244x_{5n+6}+155x_{3n+5}-1220x_{3n+4}+310x_{n+3}-2440x_{n+2}\right]$
* $\frac{1}{8}\left[31y_{5n+5}-15x_{5n+6}+155y_{3n+3}-75x_{3n+4}+310y_{n+1}-150x_{n+2}\right]$
* $\frac{1}{2}\left[31y_{5n+6}-120x_{5n+6}+155y_{3n+4}-600x_{3n+4}+310y_{n+2}-1200x_{n+2}\right]$
* $\frac{1}{8}\left[31y_{5n+7}-945x_{5n+6}+155y_{3n+5}-4725x_{3n+4}+310y_{n+3}-9450x_{n+2}\right]$
* $\frac{1}{62}\left[244y_{5n+5}-15x_{5n+7}+1220y_{3n+3}-75x_{3n+5}+2440y_{n+1}-150x_{n+3}\right]$
* $\frac{1}{2}\left[61y_{5n+6}-30x_{5n+7}+305y_{3n+4}-150x_{3n+5}+610y_{n+2}-300x_{n+3}\right]$
* $\frac{1}{2}\left[8y_{5n+5}-y_{5n+6}+40y_{3n+3}-5y_{3n+4}+80y_{n+1}-10y_{n+2}\right]$
* $\frac{1}{16}\left[63y_{5n+5}-y_{5n+7}+315y_{3n+3}-5y_{3n+5}+630y_{n+1}-10y_{n+3}\right]$
* $\frac{1}{2}\left[63y_{5n+6}-8y_{5n+7}+315y_{3n+4}-40y_{3n+5}+630y_{n+2}-80y_{n+3}\right]$
* $\frac{1}{2}\left[8y_{5n+5}-y_{5n+6}+40y_{3n+3}-5y_{3n+4}+80y_{n+1}-10y_{n+2}\right]$

REMARKABLE OBSERVATIONS:

I. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of Hyperbolas which are presented in Table2 below:

Table2: Hyperbolas

S.No	Hyperbola	(X,Y)
1	$15X^2 - Y^2 = 240$	$(4x_{n+2} - 31x_{n+1}, 120x_{n+1} - 15x_{n+2})$
2	$240X^2 - Y^2 = 15360$	$(x_{n+3} - 61x_{n+1}, 945x_{n+1} - 15x_{n+3})$
3	$15X^2 - Y^2 = 240$	$(4y_{n+1} - 15x_{n+1}, 60x_{n+1} - 15y_{n+1})$
4	$240X^2 - Y^2 = 3840$	$(y_{n+2} - 30x_{n+1}, 465x_{n+1} - 15y_{n+2})$
5	$15X^2 - 4Y^2 = 922560$	$(8y_{n+3} - 1890x_{n+1}, 465x_{n+1} - 15y_{n+2})$
6	$15X^2 - Y^2 = 240$	$(31x_{n+3} - 244x_{n+2}, 945x_{n+2} - 120x_{n+3})$
7	$15X^2 - 16Y^2 = 3840$	$(31y_{n+1} - 15x_{n+2}, 15x_{n+2} - 30y_{n+1})$
8	$15X^2 - Y^2 = 240$	$(31y_{n+2} - 120x_{n+2}, 465x_{n+2} - 120y_{n+2})$
9	$15X^2 - 16Y^2 = 3840$	$(31y_{n+3} - 945x_{n+2}, 915x_{n+2} - 30y_{n+3})$
10	$15X^2 - Y^2 = 230640$	$(244y_{n+1} - 15x_{n+3}, 60x_{n+3} - 945y_{n+1})$
11	$240X^2 - Y^2 = 3840$	$(61y_{n+2} - 30x_{n+3}, 465x_{n+3} - 945y_{n+2})$

12	$15X^2 - Y^2 = 240$	$(244y_{n+3} - 945x_{n+3}, 3660x_{n+3} - 945y_{n+3})$
13	$15X^2 - Y^2 = 240$	$(8y_{n+1} - y_{n+2}, 4y_{n+2} - 31y_{n+1})$
14	$15X^2 - 16Y^2 = 15360$	$(63y_{n+1} - y_{n+3}, y_{n+3} - 61y_{n+1})$
15	$15X^2 - Y^2 = 240$	$(63y_{n+2} - 8y_{n+3}, 31y_{n+3} - 244y_{n+2})$

II. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of Parabolas which are presented in Table3 below:

Table3: Parabolas

S.No	**Parabola**	(X, Y)
1	$30X - Y^2 = 240$	$(4x_{2n+3} - 31x_{2n+2}, 120x_{n+1} - 15x_{n+2})$
2	$960X - Y^2 = 15360$	$(x_{2n+4} - 61x_{2n+2}, 945x_{n+1} - 15x_{n+3})$
3	$30X - Y^2 = 240$	$(4y_{2n+2} - 15x_{2n+2}, 60x_{n+1} - 15y_{n+1})$
4	$480X - Y^2 = 3840$	$(y_{2n+3} - 30x_{2n+2}, 465x_{n+1} - 15y_{n+2})$
5	$465X - Y^2 = 230640$	$(8y_{2n+4} - 1890x_{2n+2}, 3660x_{n+1} - 15y_{n+3})$
6	$30X - Y^2 = 240$	$(31x_{2n+4} - 244x_{2n+3}, 945x_{n+2} - 120x_{n+3})$
7	$15X - 2Y^2 = 480$	$(31y_{2n+2} - 15x_{2n+3}, 15x_{n+2} - 30y_{n+1})$
8	$30X - Y^2 = 240$	$(31y_{2n+3} - 120x_{2n+3}, 465x_{n+2} - 120y_{n+2})$
9	$15X - 2Y^2 = 480$	$(31y_{2n+4} - 945x_{2n+3}, 915x_{n+2} - 30y_{n+3})$

10	$930X - Y^2 = 230640$	$(244y_{2n+2} - 15x_{2n+4}, 60x_{n+3} - 945y_{n+1})$
11	$480X - Y^2 = 3840$	$(61y_{2n+3} - 30x_{2n+4}, 465x_{n+3} - 945y_{n+2})$
12	$30X - Y^2 = 240$	$(244y_{2n+4} - 945x_{2n+4}, 3660x_{n+3} - 945y_{n+3})$
13	$30X - Y^2 = 240$	$(8y_{2n+2} - y_{2n+3}, 4y_{n+2} - 31y_{n+1})$
14	$15X - Y^2 = 960$	$(63y_{2n+2} - y_{2n+4}, y_{n+3} - 61y_{n+1})$
15	$30X - Y^2 = 240$	$(63y_{2n+3} - 8y_{2n+4}, 31y_{n+3} - 244y_{n+2})$

CONCLUSION:

In this paper, we have presented infinitely many integer solutions for the hyperbola represented by the positive Pell equation $y^2 = 15x^2 + 16$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of positive Pell equations and determine their integer solutions along with suitable properties.

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