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Observations on the Integral Solutions of Ternary Quadratic Equation $x^2+y^2=z^2+12$

A.VIJAYASANKAR[1], SHARADHAKUMAR[2], M.A. GOPALAN[3]

[1] Assistant Professor, Department of Mathematics, National College, Affiliated to Bharathidasan University, Trichy-620 001, Tamil Nadu, India.
email:avsankar70@yahoo.com,
[2] Research Scholar, Department of Mathematics, National College, Affiliated to Bharathidasan University, Trichy- 620 001, Tamil Nadu, India.
email : sharadhak12@gmail.com
[3] Professor, Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Trichy-620 002, Tamil Nadu, India.
email: mayilgopalan@gmail.com

Abstract

This paper illustrates the process of obtaining different sets of non-zero distinct integer solutions to the non-homogeneous ternary quadratic Diophantine equation given by $x^2+y^2=z^2+12$.

Keywords: non-homogeneous quadratic , ternary quadratic, integer solutions

Introduction

It is known that Diophantine equations with multidegree and multiple variables are rich in variety[1,2].

While searching for the collection of second degree equations with three unknowns,the authors came across the paper s[3,4,5,6,7]in which the authors obtained integer solutions to the ternary quadratic equations $x^2+y^2=z^2+N, N=1,\pm4,8.$ entitled " Integral Solutions of Ternary Quadratic Equation $x^2+y^2=z^2-4$ "

The above papers motivated us for obtaining non zero distinct integer solutions to the above equation for other values to N. This communication illustrates process of obtaining different sets of non-zero distinct integer solutions to the non-homogeneous ternary quadratic Diophantine equation given by $x^2+y^2=z^2+12$.

Method of analysis

The non-homogeneous ternary quadratic Diophantine equation under consideration is

$$x^2 + y^2 = z^2 + 12 \qquad (1)$$

It is observed that (1) is satisfied by the following triples:

$(x, y, z) = (6,1,5), (3,2,1), (2,3,-1)(1,6,-5), (-1,-6,5), (-2,-3,1)(-3,-2,-1), (-6,-1,-5),$
$(6,1,-5), (3,2,-1), (2,3,1), (1,6,5), (-6,-1,5), (-3,-2,1), (-2,-3,-1), (-1,-6,-5),$
$(6,-1,5), (-1,6,5), (-3,2,-1), (-2,3,1)$

The process of obtaining different sets of integer solutions to (1) is illustrated below:

Illustration 1:

The substitution of the linear transformations

$$z = u + v\ , x = u - v, u \neq v \neq 0 \qquad (2)$$

in (1) leads to

$$y^2 = 4(uv + 3) \qquad (3)$$

Remember that u, v are non-zero distinct integers and it is possible to choose them such that the R.H.S. of (3) is a perfect square and the value of y is obtained. Substituting the values of u, v in (2),the corresponding integer values to x, z are determined .A few numerical examples are exhibited in Table 1 below:

Table 1 : Examples

u	v	x	y	z
k^2-3	1	k^2-4	$2k$	k^2-2
$2k^2-2k-1$	2	$2k^2-2k-3$	$4k-2$	$2k^2-2k+1$
$3k^2-1$	3	$3k^2-4$	$6k$	$3k^2+2$
$6k^2-6k+1$	6	$6k^2-6k-5$	$12k-6$	$6k^2-6k+7$
$\frac{(3k^2+k+4)^2-12}{52}$	13	$\frac{(3k^2+k+4)^2-12}{52}-13$	$3k^2+k+4$	$\frac{(3k^2+k+4)^2-12}{52}+13$
$22k^2-34k+13$ $22k^2-10k+1$	22	$22k^2-34k-9$ $22k^2-10k-21$	$44k-34$ $44k-10$	$22k^2-34k+35$ $22k^2-34k+23$

Illustration 2:

The choice

$$z = x + h, h \geq 0 \tag{4}$$

in (1) leads to the parabola

$$y^2 = 2hx + h^2 + 12 \tag{5}$$

It is possible to choose h, x so that the R.H.S. of (5) is a perfect square and the value of y is obtained. Substituting the values of h, x in (4),the corresponding value of z satisfying (1) is obtained. For simplicity and brevity ,a few examples are given in Table 2 below:

Table 2 : Examples

h	x	y	z
1	$2k^2+6k-2$	$2k+3$	$2k^2+6k-1$
2	k^2+4k	$2k+4$	k^2+4k+2
3	$6k^2+6k-2$	$6k+3$	$6k^2+6k+1$
4	$2k^2+2k-3$	$4k+2$	$2k^2+2k+1$
6	$3k^2+6k-1$	$6k+6$	$3k^2+6k+5$
11	$88k^2-84k+14$ $22k^2-2k-6$	$44k-21$ $22k-1$	$88k^2-84k+25$ $22k^2-2k+5$

Illustration 3 :

The substitution of the linear transformation

$$z = k\,x,(k \geq 1)$$

(6)

in (1) leads to the positive pell equation

$$y^2 = (k^2 - 1)x^2 + 12$$

(7)

which is solvable for special values to k. For example ,considering the value of k to be 5 in (7), one obtains the positive pell equation

$$y^2 = 24x^2 + 12$$

(8)

Employing the standard procedure ,the corresponding integer solutions to (8) are given by

$$x_{n+1} = \frac{f_n}{2} + \frac{\sqrt{6}\,g_n}{4}\ ,y_{n+1} = 3f_n + \sqrt{6}\,g_n$$

(9)

where

$$f_n = (5 + 2\sqrt{6})^{n+1} + (5 - 2\sqrt{6})^{n+1}\ ,g_n = (5 + 2\sqrt{6})^{n+1} - (5 - 2\sqrt{6})^{n+1}$$

In view of (6), it is seen that

$$z_{n+1} = 5(\frac{f_n}{2} + \frac{\sqrt{6}\,g_n}{4})$$

(10)

Thus,(9) and (10) represent the integer solutions to (1).

Illustration 4:

The substitution of the linear transformations

$$z=(k+1)\alpha\ ,x=k\alpha \tag{11}$$

in (1) leads to the positive pell equation

$$y^2=(2k+1)\alpha^2+12 \tag{12}$$

for which the integer solutions exist when k takes particular values. For example, considering the value of k to be 6 in (12),it gives the positive pell equation

$$y^2=13\alpha^2+12 \tag{13}$$

After some algebra ,the corresponding integer solutions to (13) are given by

$$y_{n+1}=\frac{5}{2}f_n+\frac{\sqrt{13}}{2}g_n \tag{14}$$

$$\alpha_{n+1}=\frac{f_n}{2}+\frac{5\sqrt{13}\,g_n}{26} \tag{15}$$

where

$$f_n=(649+180\sqrt{13})^{n+1}+(649-180\sqrt{13})^{n+1}\ ,g_n=(649+180\sqrt{13})^{n+1}-(649-180\sqrt{13})^{n+1}$$

Using (15) in (11), one obtains that

$$x_{n+1}=6\left(\frac{f_n}{2}+\frac{5\sqrt{13}\,g_n}{26}\right),z_{n+1}=7\left(\frac{f_n}{2}+\frac{5\sqrt{13}\,g_n}{26}\right), \tag{16}$$

Thus,(14) and (16) represent the integer solutions to (1).

Illustration 5:

Assume

$$x=y+h,h\geq 0$$

(17)

Using (17) in (1) and performing a few calculations, one arrives at the pell equation

$$Y^2 = 2z^2 - (h^2 - 24)$$

(18)

where

$$Y = 2y + h$$

(19)

To obtain the integer solutions to (18), one has to take particular values to h.

Example :

The choice $h = 1$ in (18) gives the equation

$$Y^2 = 2z^2 + 23$$

Employing the standard procedure and performing some algebra ,the corresponding integer solutions to (1) are found to be

$$x_{n+1} = \frac{5f_n + \sqrt{2}g_n + 2}{4}$$
$$y_{n+1} = \frac{5f_n + \sqrt{2}g_n - 2}{4}$$
$$z_{n+1} = \frac{2f_n + 5\sqrt{2}g_n}{4}$$

where

$$f_n = (3+2\sqrt{2})^{n+1} + (3-2\sqrt{2})^{n+1}$$
$$g_n = (3+2\sqrt{2})^{n+1} - (3-2\sqrt{2})^{n+1}$$

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