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OBSERVATION ON THE INTEGER SOLUTION OF THE POSITIVE PELL EQUATION

$$y^2 = 15x^2 + 21$$

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Abstract :

The binary quadratic equation $y^2 = 15x^2 + 21$ is considered and a few interesting properties among the solutions are presented. Employing the integral solutions of the equation under consideration, a few remarkable observations are illustrated.

Keywords: Binary quadratic, Hyperbola, Integral solutions, Pell equation.

Introduction:

Any non-homogeneous binary quadratic equation of the form $y^2 - Dx^2 = 1$, where D is a given positive non square integer, requiring integer solutions for $x \,\&\, y$ is called Pellian equation (also known pell-fermat equation). In cartesian co-ordinates, the equation has the form of a hyperbola. The pellian equation has infinitely many distinct integer solutions as long as D is not a perfect square and the solutions are easily generated recursively from a single fundamental solution, namely, the solution with x, y positive integers of smallest possible size. One may refer [1-9] for a few choices of pellian equations along with their corresponding integer solutions.

The solutions to Pellian equations have long been of interest to mathematicians. Even small values of D can lead to fundamental solutions which are quite large. For example, when D=61, the fundamental solution is (1766319049, 226153980). The above results motivated us to search for integer solutions to other choices of Pellian equation. This paper concerns with the Pellian equation $y^2=15x^2+21$, a few interesting properties among the solutions are presented. Employing the integral solutions of the equation under consideration, a few remarkable observations are illustrated.

METHOD OF ANALYSIS

The Diophantine equation representing the binary quadratic to be solved for its non-zero distinct integer solution is

$$y^2 = 15x^2 + 21 \tag{1}$$

whose smallest positive integer solution is $X_0=1, Y_0 = 6$

To obtain the other solutions (1), consider the pell equation

$$Y^2 = 15X^2 + 1 \tag{2}$$

Whose smallest positive integer solution is $\left(\tilde{X}_0, \tilde{Y}_0\right) = (1,4)$

If $\left(\tilde{X}_n, \tilde{Y}_n\right)$ represents the general solution of (2) is given by

$$\tilde{X}_n = \frac{1}{2\sqrt{15}} g_n$$

$$\tilde{Y}_n = \frac{1}{2} f_n$$

Where

$$f_n = \left(4+\sqrt{15}\right)^{n+1} + \left(4-\sqrt{15}\right)^{n+1}$$

$$g_n = \left(4+\sqrt{15}\right)^{n+1} - \left(4-\sqrt{15}\right)^{n+1}$$

Applying Brahmagupta lemma between (X_0, Y_0) and $(\tilde{X}_n, \tilde{Y}_n)$ we have

$$X_{n+1} = Y_0\tilde{X}_n + X_0\tilde{Y}_n$$

$$Y_{n+1} = Y_0\tilde{Y}_n + DX_0\tilde{X}_n$$

$$\Rightarrow X_{n+1} = \frac{3}{\sqrt{15}} g_n + \frac{1}{2} f_n \tag{3}$$

$$\Rightarrow Y_{n+1} = 3f_n + \frac{15}{2\sqrt{15}} g_n \tag{4}$$

Substituting (3) & (4) in (2), the other integer solutions of (1) are given by

$$2\sqrt{15}x_{n+1} = \sqrt{15} f_n + 6g_n \tag{5}$$

$$2\sqrt{15}y_{n+1} = 6\sqrt{15} f_n + 15g_n \tag{6}$$

Replace n by $n+1$ in (5) we get

$$2\sqrt{15}x_{n+2} = \sqrt{15}\left(4f_n + \sqrt{15}g_n\right) + 6\left(4g_n + \sqrt{15} f_n\right)$$

$$2\sqrt{15}x_{n+2} = 10\sqrt{15} f_n + 39g_n \tag{7}$$

Replace n by $n+1$ in (7) we get

$$2\sqrt{15}x_{n+3} = 10\sqrt{15}\left(4f_n + \sqrt{15}g_n\right) + 39\left(4g_n + \sqrt{15} f_n\right)$$
$$2\sqrt{15}x_{n+3} = 79\sqrt{15} f_n + 306g_n \tag{8}$$

Eliminating f_n, g_n between (5),(7) &(8) we have

$$x_{n+1} - 8x_{n+2} + x_{n+3} = 0, n = -1,0,1,2..... \tag{9}$$

In a similar manner, one obtains

$$2\sqrt{15}y_{n+2} = 39\sqrt{15} f_n + 150g_n \tag{10}$$

$$2\sqrt{15}y_{n+3} = 306\sqrt{15} f_n + 1185g_n \tag{11}$$

Eliminating f_n, g_n between (6),(10) &(11) we have

$$y_{n+1} - 8y_{n+2} + y_{n+3} = 0, \; n = -1,0,1,2..... \quad (12)$$

Thus (9) & (12) represent the recurrence relations satisfied by the value of x & y respectively.

Some numerical examples of x_n & y_n satisfying (1) are given the table 1 below :

n	x_{n+1}	y_{n+1}
-1	1	6
0	10	39
1	79	306
2	622	2409
3	4897	18966
4	38554	149319

Observations:

- The pair of (x_n , y_n) has different parity.
- $x_{2n-1} \equiv 0 \pmod{2}, y_{2n} \equiv 0 \pmod{2}$
- $y_{n+1} \equiv 0 \pmod{3}$

1.A few interesting relations among the solutions are given below:

- consider (5)&(7) we get

  $$2\sqrt{15}x_{n+1} = \sqrt{15} f_n + 6g_n$$

$$2\sqrt{15}x_{n+2} = 10\sqrt{15}f_n + 39g_n$$

Solving above two equations we get,

$$f_n = \frac{2}{21}\left(6x_{n+2} - 39x_{n+1}\right)$$

$$g_n = \frac{2}{21}\sqrt{15}\left(10x_{n+1} - x_{n+2}\right)$$

Substitute f_n & g_n value in (6), we get

$$\therefore\ x_{n+2} - 4x_{n+1} - y_{n+1} = 0$$

For Simplicity and clear understanding, the other choices of interesting relations are given below,

- $4x_{n+2} - x_{n+1} - y_{n+2} = 0$
- $31x_{n+2} - 4x_{n+1} - y_{n+3} = 0$
- $21x_{n+3} - 3099x_{n+1} - 168y_{n+1} = 0$
- $21x_{n+3} - 651x_{n+1} - 168y_{n+1} = 0$
- $84x_{n+3} - 84x_{n+1} - 168y_{n+2} = 0$
- $65x_{n+3} - 21x_{n+1} - 168y_{n+3} = 0$
- $168x_{n+3} - 1302x_{n+2} - 42y_{n+1} = 0$
- $42x_{n+3} - 168x_{n+2} - 42y_{n+2} = 0$
- $168x_{n+3} - 42x_{n+2} - 42y_{n+3} = 0$
- $168y_{n+1} + 42y_{n+2} - 630x_{n+1} = 0$
- $168y_{n+2} - 42y_{n+1} - 630x_{n+2} = 0$
- $1302y_{n+2} - 168y_{n+1} - 630x_{n+3} = 0$
- $168y_{n+3} - 1302y_{n+2} - 630x_{n+1} = 0$
- $168y_{n+2} - 42y_{n+3} + 630x_{n+2} = 0$
- $168y_{n+3} - 42y_{n+2} - 630x_{n+3} = 0$
- $1716y_{n+3} - 81636y_{n+1} - 67410x_{n+2} = 0$
- $10419y_{n+3} - 472569y_{n+1} - 67410x_{n+1} = 0$
- $81636y_{n+3} - 3698916y_{n+1} - 67410x_{n+1} = 0$

2.Each of the following expressions representing Perfect Square :

- consider (5)&(7) we get

$$2\sqrt{15}x_{n+1} = \sqrt{15} f_n + 6g_n$$

$$2\sqrt{15}x_{n+2} = 10\sqrt{15} f_n + 39g_n$$

Solving above two equations we get,

$$f_n = \frac{2}{7}\left(2x_{n+2} - 13x_{n+1}\right)$$

$$g_n = \frac{2}{21}\sqrt{15}\left(10x_{n+1} - x_{n+2}\right)$$

Replace n by $2n+1$ in f_n we get

$$f_{2n+1} = \frac{2}{7}\left[2x_{2n+3} - 13x_{2n+2}\right]$$

We know that, $f_{2n+1} + 2 = f_n^{\ 2}$

$$\Rightarrow \frac{1}{7}\left[4x_{2n+3} - 26x_{2n+2} + 14\right] = f_n^{\ 2}$$

$\therefore \frac{1}{7}\left[4x_{2n+3} - 26x_{2n+2} + 14\right]$ is a Perfect square.

For Simplicity and clear understanding, the other choices of perfect square are presented below :

- $\frac{1}{14}\left[x_{2n+4} - 51x_{2n+2} + 28\right]$
- $\frac{2}{7}\left[2y_{2n+2} - 5x_{2n+2} + 7\right]$
- $\frac{1}{7}\left[y_{2n+3} - 25x_{2n+2} + 14\right]$
- $\frac{1}{651}\left[12y_{2n+4} - 2370x_{2n+2} + 1302\right]$
- $\frac{2}{7}\left[13x_{2n+4} - 102x_{2n+3} + 14\right]$
- $\frac{1}{14}\left[13y_{2n+2} - 5x_{2n+3} + 28\right]$
- $\frac{2}{7}\left[13y_{2n+3} - 50x_{2n+4} + 7\right]$
- $\frac{1}{14}\left[13y_{2n+4} - 395x_{2n+3} + 28\right]$

- $\frac{1}{217}\left[102y_{2n+2} - 5x_{2n+4} + 434\right]$
- $\frac{1}{7}\left[51y_{2n+3} - 25x_{2n+4} + 14\right]$
- $\frac{2}{7}\left[102y_{2n+4} - 395x_{2n+4} + 7\right]$
- $\frac{2}{21}\left[10y_{2n+2} - y_{2n+3} + 21\right]$
- $\frac{1}{84}\left[79y_{2n+2} - y_{2n+4} + 168\right]$
- $\frac{2}{21}\left[79y_{2n+3} - 10y_{2n+4} + 21\right]$

3. Each of the following expressions representing cubical integers:

- consider (5)&(7) we get

$$2\sqrt{15}x_{n+1} = \sqrt{15}f_n + 6g_n$$

$$2\sqrt{15}x_{n+2} = 10\sqrt{15}f_n + 39g_n$$

Solving above two equations we get,

$$f_n = \frac{2}{7}\left(2x_{n+2} - 13x_{n+1}\right)$$

$$g_n = \frac{2}{21}\sqrt{15}\left(10x_{n+1} - x_{n+2}\right)$$

Replace n by $3n+2$ in f_n we get

$$f_{3n+2} = \frac{2}{7}\left[2x_{3n+4} - 13x_{3n+3} + 6x_{2n+2} - 39x_{n+1}\right]$$

We know that, $f_{3n+2} + 3f_n = f_n^{\ 3}$

$\therefore\ f_{3n+2} = \frac{2}{7}\left[2x_{3n+4} - 13x_{3n+3} + 6x_{n+2} - 39x_{n+1}\right]$ is a cubical integer.

For Simplicity and clear understanding, the other choices of cubical integer are presented below:

- $\frac{1}{14}\left[x_{3n+5} - 51x_{3n+3} + 3x_{n+3} - 153x_{n+1}\right]$

- $\frac{2}{7}\left[2y_{3n+3}-5x_{3n+3}+6y_{n+1}-15x_{n+1}\right]$
- $\frac{1}{7}\left[y_{3n+4}-25x_{3n+3}+3y_{n+2}-75x_{n+1}\right]$
- $\frac{1}{651}\left[12y_{3n+5}-2370x_{3n+3}+36y_{3n+3}-7110x_{n+1}\right]$
- $\frac{2}{7}\left[13x_{3n+5}-102x_{3n+4}+39x_{n+3}-306x_{n+2}\right]$
- $\frac{1}{14}\left[13y_{3n+3}-5x_{3n+4}+39y_{n+1}-15x_{n+2}\right]$
- $\frac{2}{7}\left[13y_{3n+4}-50x_{3n+5}+39y_{n+2}-150x_{n+2}\right]$
- $\frac{1}{14}\left[13y_{3n+5}-395x_{3n+4}+39y_{n+3}-1185x_{n+2}\right]$
- $\frac{2}{217}\left[102y_{3n+3}-5x_{3n+5}+612y_{n+1}-30x_{n+3}\right]$
- $\frac{1}{7}\left[51y_{3n+4}-25x_{3n+5}+153y_{n+2}-75x_{n+3}\right]$
- $\frac{2}{7}\left[102y_{3n+5}-395x_{3n+5}+306y_{n+3}-1185x_{n+3}\right]$
- $\frac{2}{21}\left[10y_{3n+3}-y_{3n+4}+30y_{n+1}-3y_{n+2}\right]$
- $\frac{1}{84}\left[79y_{3n+3}-y_{3n+5}+237y_{n+1}-3y_{n+2}\right]$
- $\frac{2}{21}\left[79y_{3n+4}-10y_{3n+5}+237y_{n+2}-30y_{n+3}\right]$

4. Each of the following expressions representing Biquadratic integers:

- consider (5)&(7) we get

$$2\sqrt{15}x_{n+1}=\sqrt{15}f_n+6g_n$$

$$2\sqrt{15}x_{n+2}=10\sqrt{15}f_n+39g_n$$

Solving above two equations we get,

$$f_n=\frac{2}{7}\left(2x_{n+2}-13x_{n+1}\right)$$

$$g_n=\frac{2}{21}\sqrt{15}\left(10x_{n+1}-x_{n+2}\right)$$

Replace n by $4n+3$ in f_n we get

$$f_{4n+3} = \frac{2}{7}\left[2x_{4n+5} - 13x_{4n+4} + 8x_{2n+3} - 52x_{n+2} + 21\right]$$

We know that, $f_{3n+2} + 3f_n = f_n^{\,3}$

$\therefore \frac{2}{7}\left[2x_{4n+5} - 13x_{4n+4} + 8x_{2n+3} - 52x_{n+2} + 21\right]$ is a biquadratic integer.

For Simplicity and clear understanding, the other choices of interesting biquadratic integers are presented below :

- $\frac{1}{14}\left[x_{4n+6} - 51x_{4n+4} + 4x_{2n+4} - 204x_{2n+2} + 84\right]$
- $\frac{2}{7}\left[2y_{4n+4} - 5x_{4n+4} + 8y_{2n+2} - 20x_{2n+2} + 21\right]$
- $\frac{1}{7}\left[y_{4n+5} - 25x_{4n+4} + 4y_{2n+3} - 100x_{2n+2} + 42\right]$
- $\frac{1}{651}\left[12y_{4n+6} - 2370x_{4n+4} + 48y_{2n+4} - 9480x_{2n+2} + 3906\right]$
- $\frac{2}{7}\left[13x_{4n+6} - 102x_{4n+5} + 52x_{2n+4} - 408x_{2n+3} + 49\right]$
- $\frac{1}{14}\left[13y_{4n+4} - 5x_{4n+5} + 52y_{2n+2} - 20x_{2n+3} + 84\right]$
- $\frac{2}{7}\left[13y_{4n+5} - 50x_{4n+6} + 52y_{2n+3} - 200x_{2n+4} + 49\right]$
- $\frac{1}{14}\left[13y_{4n+6} - 395x_{4n+5} + 52y_{2n+4} - 1580x_{2n+3} + 84\right]$
- $\frac{2}{217}\left[102y_{4n+4} - 5x_{4n+6} + 408y_{2n+2} - 20x_{2n+4} + 1519\right]$
- $\frac{1}{7}\left[51y_{4n+5} - 25x_{4n+6} + 204y_{2n+3} - 100x_{2n+4} + 42\right]$
- $\frac{2}{7}\left[102y_{4n+6} - 395x_{4n+6} + 408y_{2n+4} - 1580x_{2n+4} + 21\right]$
- $\frac{2}{21}\left[10y_{4n+4} - y_{4n+5} + 40y_{2n+2} - 4y_{2n+3} + 63\right]$
- $\frac{1}{84}\left[79y_{4n+4} - y_{4n+6} + 316y_{2n+2} - 4y_{2n+4} + 504\right]$

- $\frac{2}{21}\left[79y_{4n+5}-10y_{4n+6}+316y_{2n+3}-40y_{2n+4}+63\right]$

5. Employing linear combinations among the solutions, one obtains solutions to other choices of hyperbolas

illustration :

Solving (5) & (7) we get,

$$f_n=\frac{2}{7}\left(2x_{n+2}-13x_{n+1}\right)$$

$$g_n=\frac{2}{21}\sqrt{15}\left(10x_{n+1}-x_{n+2}\right)$$

We know that ,

$$f_n^{\,2}-g_n^{\,2}=4$$

Let $X=2x_{n+2}-13x_{n+1}$, $Y=10x_{n+1}-x_{n+2}$ in f_n & g_n we get

$$15Y^2=X^2-441$$

Note that (X, Y) satisfies the hyperbola.

For Simplicity and clear understanding, the other choices of interesting hyperbolas are presented below :

Choice : 1

Let $X=x_{n+3}-51x_{n+1}\,Y=79x_{n+1}-x_{n+3}$

$$\Rightarrow 15Y^2=36X^2-28224$$

Note that (X, Y) satisfies the hyperbola.

Choice : 2

Let $X=2y_{n+1}-5x_{n+1}$, $Y=6x_{n+1}-y_{n+1}$

$$\Rightarrow 15Y^2=9X^2-441$$

Note that (X, Y) satisfies the hyperbola.

Choice : 3

Let $X = y_{n+2} - 25x_{n+1}, Y = 39x_{n+1} - y_{n+2}$

$\Rightarrow 15Y^2 = 36X^2 - 7056$

Note that (X, Y) satisfies the hyperbola.

Choice : 4

Let $X = 12y_{n+3} - 2370x_{n+1}, Y = 306x_{n+1} - y_{n+3}$

$\Rightarrow 60Y^2 = X^2 - 1695204$

Note that (X, Y) satisfies the hyperbola.

Choice : 5

Let $X = 13x_{n+3} - 102x_{n+2}, Y = 79x_{n+2} - 10x_{n+3}$

$\Rightarrow 15Y^2 = 9X^2 - 441$

Note that (X, Y) satisfies the hyperbola.

Choice : 6

Let $X = 13y_{n+1} - 5x_{n+2}, Y = 3x_{n+2} - 5y_{n+1}$

$\Rightarrow 20Y^2 = 3X^2 - 2352$

Note that (X, Y) satisfies the hyperbola.

Choice : 7

Let $X = 13y_{n+2} - 50x_{n+2}, Y = 39x_{n+2} - 10y_{n+2}$

$\Rightarrow 15Y^2 = 9X^2 - 441$

Note that (X, Y) satisfies the hyperbola.

Choice : 8

8Let $X = 13y_{n+3} - 395x_{n+2}, Y = 153x_{n+2} - 5y_{n+3}$

$\Rightarrow 20Y^2 = 3X^2 - 2352$

Note that (X, Y) satisfies the hyperbola.

Choice : 9

Let $X = 102y_{n+1} - 5x_{n+3}$, $Y = 6x_{n+3} - 79y_{n+1}$

$\Rightarrow 15Y^2 = 9X^2 - 423801$

Note that (X, Y) satisfies the hyperbola.

Choice : 10

Let $X = 51y_{n+2} - 25x_{n+3}$, $Y = 39x_{n+3} - 79y_{n+2}$

$\Rightarrow 15Y^2 = 36X^2 - 7056$

Note that (X, Y) satisfies the hyperbola.

Choice : 11

Let $X = 102y_{n+3} - 395x_{n+3}$, $Y = 306x_{n+3} - 79y_{n+3}$

$\Rightarrow 15Y^2 = 9X^2 - 441$

Note that (X, Y) satisfies the hyperbola.

Choice : 12

Let $X = 10y_{n+1} - y_{n+2}$, $Y = 2y_{n+2} - 13y_{n+1}$

$\Rightarrow 3Y^2 = 5X^2 - 2205$

Note that (X, Y) satisfies the hyperbola.

Choice : 13

Let $X = 79y_{n+1} - y_{n+3}$, $Y = y_{n+3} - 51y_{n+1}$

$\Rightarrow 588Y^2 = 245X^2 - 6914880$

Note that (X, Y) satisfies the hyperbola.

Choice : 14

Let $X = 79y_{n+2} - 10y_{n+3}$, $Y = 13y_{n+3} - 102y_{n+2}$

$\Rightarrow 3Y^2 = 5X^2 - 2205$

Note that (X, Y) satisfies the hyperbola.

6. Employing linear combinations among the solutions, one obtains solutions to other choices of parabolas:

Illustration :

Solving (5) & (7) we get,

$$f_n = \frac{2}{7}(2x_{n+2} - 13x_{n+1})$$

$$g_n = \frac{2}{21}\sqrt{15}(10x_{n+1} - x_{n+2})$$

We know that ,

$$f_n^{\ 2} - g_n^{\ 2} = 4$$

Let $X = 4x_{2n+3} - 26x_{2n+2}$, $Y = 10x_{n+1} - x_{n+2}$ in f_n & g_n we get

$$20Y^2 = 21X - 294$$

Note that (X, Y) satisfies the parabola.

For Simplicity and clear understanding, the other choices of interesting parabolas are presented below:

Choice : 1

Let $X = x_{2n+4} - 51x_{2n+2}$, $Y = 79x_{n+1} - x_{n+3}$

$$\Rightarrow 15Y^2 = 504X - 14112$$

Note that (X, Y) satisfies the parabola.

Choice : 2

Let $X = 2y_{2n+2} - 5x_{2n+2}$, $Y = 6x_{n+1} - y_{n+1}$

$$\Rightarrow 30Y^2 = 63X - 441$$

Note that (X, Y) satisfies the parabola.

Choice : 3

Let $X = y_{2n+3} - 25x_{2n+2}$, $Y = 39x_{n+1} - y_{n+2}$

$\Rightarrow 15Y^2 = 252X - 3528$

Note that (X, Y) satisfies the parabola.

Choice : 4

Let $X = 12y_{2n+4} - 2370x_{2n+2}, Y = 306x_{n+1} - y_{n+3}$

$\Rightarrow 60Y^2 = 651X - 847602$

Note that (X, Y) satisfies the parabola.

Choice : 5

Let $X = 13x_{2n+4} - 102x_{2n+3}, Y = 79x_{n+2} - 10x_{n+3}$

$\Rightarrow 30Y^2 = 63X - 441$

Note that (X, Y) satisfies the parabola.

Choice : 6

Let $X = 13y_{2n+2} - 5x_{2n+3}, Y = 3x_{n+2} - 5y_{n+1}$

$\Rightarrow 10Y^2 = 21X - 588$

Note that (X, Y) satisfies the parabola.

Choice : 7

Let $X = 13y_{2n+3} - 50x_{2n+3}, Y = 39x_{n+2} - 10y_{n+2}$

$\Rightarrow 10Y^2 = 21X - 147$

Note that (X, Y) satisfies the parabola.

Choice : 8

Let $X = 13y_{2n+4} - 395x_{2n+3}, Y = 153x_{n+2} - 5y_{n+3}$

$\Rightarrow 10Y^2 = 21X - 588$

Note that (X, Y) satisfies the parabola.

Choice : 9

Let $X = 102y_{2n+2} - 5x_{2n+4}, Y = 6x_{n+3} - 79y_{n+1}$

$\Rightarrow 30Y^2 = 1953X - 423801$

Note that (X, Y) satisfies the pararbola.

Choice : 10

Let $X = 51y_{2n+3} - 25y_{2n+4}, Y = 39x_{n+3} - 79y_{n+2}$

$\Rightarrow 15Y^2 = 252X - 3528$

Note that (X, Y) satisfies the parabola.

Choice : 11

Let $X = 102y_{2n+4} - 395x_{2n+4}, Y = 306x_{n+3} - 79y_{n+3}$

$\Rightarrow 30Y^2 = 63X - 441$

Note that (X, Y) satisfies the parabola.

Choice : 12

Let $X = 10y_{2n+2} - y_{2n+3}, Y = 2y_{n+2} - 13y_{n+1}$

$\Rightarrow 6Y^2 = 105X - 2205$

Note that (X, Y) satisfies the parabola.

Choice : 13

Let $X = 79y_{2n+2} - y_{2n+4}, Y = y_{n+3} - 51y_{n+1}$

$\Rightarrow 3Y^2 = 105X - 17640$

Note that (X, Y) satisfies the parabola.

Choice : 14

Let $X = 79y_{2n+3} - 10y_{2n+4}, Y = 13y_{n+3} - 102y_{n+2}$

$\Rightarrow 6Y^2 = 105X - 2205$

Note that (X, Y) satisfies the parabola.

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