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The Fibonacci Series in Geometric Order

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A B S T R A C T

Leonardo of Pisa, better known as Fibonacci, first introduced the Fibonacci numbers in his book Liber abaci in 1202. The Fibonacci number sequence was mentioned in the answer to the following problem. "A man placed a pair of rabbits in an area surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if each pair begets a new pair that becomes productive from the second month on?".Higher order Fibonacci series is an extension of existing Fibonacci series that demonstrates an interesting pattern of the 'golden ratio' of the higher order Fibonacci series as shown

Keywords: Fibonacci sequence, Series & Golden Ratio.

Introduction

Sequence** [2]: A sequence of real numbers is any function $\boldsymbol{s} : \mathbb{N} \to \mathbb{R}$**

Fibonacci sequence: [1-4]

The Fibonacci sequence is one of the most famous — and widely written about — number sequences in mathematics. So why write about it if so much has already been written about it? Because I can and because it's enjoyable — there's a reason it's so popular! I intend to spend the next five posts (at the very least) discussing the Fibonacci sequence and its applications in other areas of mathematics.

The Fibonacci sequence is defined by the recurrence relation

$$\mathrm{F}_1 = \mathrm{F}_2 = 1, \mathrm{F}_n = \mathrm{F}_{n-1} + \mathrm{F}_{n-2}\ (n > 2).$$

This means that the first two Fibonacci numbers are both equal to 1, and any Fibonacci number after the third can be obtained by adding the previous two Fibonacci numbers. So the third Fibonacci number is 1 + 1 = 2; the fourth is 1 + 2 = 3; and so on. The first fifteen Fibonacci numbers are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377,610.

As you can see, they grow quite quickly. The 31st Fibonacci number is the first to exceed one million, and the 45th number has already surpassed one billion.

The Geometric Fibonacci series is defined by the recurrence relation

$$\mathrm{F}_1 = 1, \mathrm{F}_2 = 2, \dots, \mathrm{F}_n = n, and\ \ \mathrm{F}_{n+1} = \sum_{r=1}^{n} \mathrm{F}_r\ \ (n > r).$$

Golden Ratio: [1-4]

The golden ratio is the division of a line segment whose total length is a + b into two parts a and b where the ratio of a + b to an equals the ratio a to b. The two ratios are roughly equal to 1.618..., which is known as the golden ratio constant and is usually denoted by Φ:

$$\frac{a+b}{a} = \frac{a}{b} = 1.618 \dots = \frac{1+\sqrt{5}}{2} = \Phi.$$

The concept of golden ratio division first appeared in art and architecture over 2400 years ago. It is possible that the magical golden ratio divisions of parts are rather closely associated with the notion of beauty in pleasing, harmonious proportions expressed by biologists, artists, musicians, historians, architects, psychologists, scientists, and even mystics in various fields of knowledge. For example, the Greek sculptor Phidias (490-430 BC) created the Parthenon statues in a way that appears to embody the golden ratio; Plato (427-347 BC) describes the five Platonic solids (the tetrahedron, cube, octahedron, dodecahedron, and icosahedron), some of which are related to the golden ratio, in his Timeous.

S. No	Conventional Fibonacci series (2 number sum)	2^2 Number sum series	2^3 Number sum series	2^4 Number sum series	2^5 Number sum series	…
1	1	1	1	1	1	…
2	1	2	2	2	2	…
3	2	3	3	3	3	…
4	3	4	4	4	4	…
5	5	10	5	5	5	…
6	8	19	6	6	6	…
7	13	36	7	7	7	…
8	21	69	8	8	8	…
9	34	134	36	9	9	…
10	55	258	71	10	10	…
11	89	497	140	11	11	…
12	144	958	277	12	12	…
13	233	1847	550	13	13	…
14	377	3560	1095	14	14	…
15	610	6862	2184	15	15	…
16	987	13227	4361	16	16	…
17	1597	25496	8714	136	17	…
18	2584	49145	17392	271	18	…
19	4181	94730	34713	540	19	…
20	6765	182598	69286	1077	20	…
21	10946	351969	138295	2150	21	…
22	17711	678442	276040	4295	22	…
23	28657	1307739	550985	8584	23	…
24	46368	2520748	1099786	17161	24	…
25	75025	4858898	2195211	34314	25	…
26	121393	9365827	4381708	68619	26	…
27	196418	18053212	8746024	137228	27	…
28	317811	34798685	17457335	274445	28	…
29	514229	67076622	34845384	548878	29	…
30	832040	129294346	69552473	1097743	30	…
31	1346269	249222865	138828906	2195472	31	…
32	2178309	480392518	277106827	4390929	32	…
33	3524578	925986351	553113868	8781842	528	…
34	5702887	1784896080	1104032525	17563548	1055	…
35	9227465	3440497814	2203683342	35126825	2108	…
36	14930352	6631772763	4398620660	70253110	4213	…
37	24157817	12783153008	8779783985	140505143	8422	…
38	39088169	24640319665	17524722586	281008136	16839	…
39	63245986	47495743250	34979892699	562011977	33672	…
40	102334155	91550988686	69820956492	1124015370	67337	…
41	165580141	176470204609	139364806157	2248013579	134666	…
42	267914296	340157256210	278176498446	4495992844	269323	…
43	433494437	655674192755	555248964367	8991917069	538636	…
44	701408733	1263852642260	1108294245392	17983696910	1077261	…
45	1134903170	2436154295834	2212189870124	35967119375	2154510	…
46	1836311903	4695838387059	4415599956263	71933689872	4309007	…
47	2971215073	9051519517908	8813675189940	143866282001	8618000	…
48	4807526976	17447364843061	17592370487181	287730368530	17235985	…
49	7778742049	33630877043862	35114920017870	575456346131	34471954	…
50	12586269025	64825599791890	70090475229583	1150903910420	68943891	…

***Although only 2^5 number series are shown here, '2^n' number of such series is possible.*

Now, if we calculate the ratio of any two numbers (such as the golden ratio for the Fibonacci series) for these '2^n' number sum series, and I denote it as $\Phi(2^n)$, it will tend to reach but never cross the '2' mark. (Except one spur)

S. No	Conventional Fibonacci Golden Ratio [Φ(2)]	Golden Ratio [$\Phi(2^2)$]	Golden Ratio [$\Phi(2^3)$]	Golden Ratio [$\Phi(2^4)$]	Golden Ratio [$\Phi(2^5)$]	…
1	1	2	2	2	2	…
2	2	1.5	1.5	1.5	1.5	…
3	1.5	1.333333333	1.33333333	1.33333333	1.33333333	…
4	1.66666667	2.5	1.25	1.25	1.25	…
5	1.6	1.9	1.2	1.2	1.2	…
6	1.625	1.894736842	1.16666667	1.16666667	1.16666667	…
7	1.61538462	1.916666667	1.14285714	1.14285714	1.14285714	…
8	1.61904762	1.942028986	4.5	1.125	1.125	…
9	1.61764706	1.925373134	1.97222222	1.11111111	1.11111111	…
10	1.61818182	1.926356589	1.97183099	1.1	1.1	…
11	1.61797753	1.927565392	1.97857143	1.09090909	1.09090909	…
12	1.61805556	1.927974948	1.98555957	1.08333333	1.08333333	…
13	1.61802575	1.927449919	1.99090909	1.07692308	1.07692308	…
14	1.61803714	1.92752809	1.99452055	1.07142857	1.07142857	…
15	1.61803279	1.927572136	1.99679487	1.06666667	1.06666667	…
16	1.61803445	1.92757239	1.99816556	8.5	1.0625	…
17	1.61803381	1.927557264	1.99586872	1.99264706	1.05882353	…
18	1.61803406	1.927561298	1.99591766	1.99261993	1.05555556	…
19	1.61803396	1.927562546	1.99596693	1.99444444	1.05263158	…
20	1.618034	1.927562186	1.99600208	1.99628598	1.05	…
21	1.61803399	1.927561802	1.99602299	1.99767442	1.04761905	…
22	1.61803399	1.927561973	1.99603318	1.99860303	1.04545455	…
23	1.61803399	1.927561998	1.99603619	1.99918453	1.04347826	…
24	1.61803399	1.927561978	1.99603468	1.99953383	1.04166667	…
25	1.61803399	1.92756197	1.99603045	1.99973772	1.04	…
26	1.61803399	1.927561976	1.99603077	1.99985427	1.03846154	…
27	1.61803399	1.927561976	1.996031	1.99991984	1.03703704	…
28	1.61803399	1.927561975	1.99603112	1.99995628	1.03571429	…
29	1.61803399	1.927561975	1.99603118	1.99997632	1.03448276	…
30	1.61803399	1.927561976	1.9960312	1.99998725	1.03333333	…
31	1.61803399	1.927561976	1.99603119	1.99999317	1.03225806	…
32	1.61803399	1.927561975	1.99603118	1.99999636	16.5	…
33	1.61803399	1.927561975	1.99603118	1.99998451	1.99810606	…
34	1.61803399	1.927561975	1.99603118	1.99998457	1.99810427	…
35	1.61803399	1.927561975	1.99603118	1.99998463	1.99857685	…
36	1.61803399	1.927561975	1.99603118	1.99998467	1.99905056	…
37	1.61803399	1.927561975	1.99603118	1.9999847	1.99940632	…
38	1.61803399	1.927561975	1.99603118	1.99998472	1.99964368	…
39	1.61803399	1.927561975	1.99603118	1.99998473	1.99979211	…
40	1.61803399	1.927561975	1.99603118	1.99998473	1.99988119	…
41	1.61803399	1.927561975	1.99603118	1.99998474	1.99993317	…
42	1.61803399	1.927561975	1.99603118	1.99998474	1.99996287	…
43	1.61803399	1.927561975	1.99603118	1.99998474	1.99997958	…
44	1.61803399	1.927561975	1.99603118	1.99998474	1.99998886	…
45	1.61803399	1.927561975	1.99603118	1.99998474	1.99999397	…
46	1.61803399	1.927561975	1.99603118	1.99998474	1.99999675	…
47	1.61803399	1.927561975	1.99603118	1.99998474	1.99999826	…
48	1.61803399	1.927561975	1.99603118	1.99998474	1.99999907	…
49	1.61803399	1.927561975	1.99603118	1.99998474	1.99999951	…

S. No	Conventional Fibonacci series (2 number sum)	3^2 Number sum series	3^3 Number sum series	3^4 Number sum series	…
1	1	1	1	81	…
2	1	2	2	188	…
3	2	3	3	3299	…
4	3	4	4	6577	…
5	5	4	5	13132	…
6	8	5	6	26241	…
7	13	6	7	52458	…
8	21	7	8	104891	…
9	34	8	9	209756	…
10	55	9	10	416542	…
11	89	30	11	832896	…
12	144	54	12	1662493	…
13	233	101	13	3318409	…
14	377	194	14	6623686	…
15	610	379	15	13221131	…
16	987	728	16	26389804	…
17	1597	1402	17	52674717	…
18	2584	2703	18	105139678	…
19	4181	5212	19	209862814	…
20	6765	10045	20	418892732	…
21	10946	19362	21	836122971	…
22	17711	37322	22	1668927533	…
23	28657	71941	23	3331231380	…
24	46368	138670	24	6649241629	…
25	75025	267295	25	13272093454	…
26	121393	515228	26	26491512191	…
27	196418	993134	27	52877884704	…
28	317811	1914327	188	105545906594	…
29	514229	3689984	356	210672920456	…
30	832040	7112673	691	420509717941	…
31	1346269	13710118	1360	839350508349	…
32	2178309	26427102	2697	1675369785318	…
33	3524578	50939877	5370	3344090329007	…
34	5702887	98189770	10715	6674908564560	…
35	9227465	189266867	21404	13323325616929	…
36	14930352	364823616	42781	26593773349154	…
37	24157817	703220130	85374	53082000791714	…
38	39088169	1355500383	170392	105953328662972	…
39	63245986	2612810996	340093	211486147608003	…
40	102334155	5036355125	678826	422132944707657	…
41	165580141	9707886634	1354955	842590519629996	…
42	267914296	18712553138	2704540	1681836948930980	…
43	433494437	36069605893	5398365	3356998989297410	…
44	701408733	69526400790	10775326	6700674652977890	…
45	1134903170	134016446455	21507871	13374755532606600	…
46	1836311903	258325006276	42930368	26696429064421500	…
47	2971215073	497937459414	85690344	53286904800180100	…
48	4807526976	959805312935	171040595	106362323452752000	…
49	7778742049	1850084225080	341402364	212302513960797000	…
50	12586269025	3566152003705	681449773	423762437401964000	…

***Although only 3^4 number series are shown here, '3^n' number of such series is possible.*

Now, if we calculate the ratio of any two numbers (such as the golden ratio for the Fibonacci series) for these '3^n' number sum series, and I denote it as $\Phi(3^n)$, it will tend to reach but never cross the '2' mark. (Except one spur)

S. No	Conventional Fibonacci Golden Ratio [$\Phi(2)$]	Golden Ratio [$\Phi(3^2)$]	Golden Ratio [$\Phi(3^3)$]	Golden Ratio [$\Phi(3^4)$]
1	1	2	2	2.320987654
2	2	1.5	1.5	17.54787234
3	1.5	1.333333333	1.333333333	1.993634435
4	1.666666667	1	1.25	1.99665501
5	1.6	1.25	1.2	1.998248553
6	1.625	1.2	1.166666667	1.999085401
7	1.615384615	1.166666667	1.142857143	1.999523428
8	1.619047619	1.142857143	1.125	1.999752124
9	1.617647059	1.125	1.111111111	1.985840691
10	1.618181818	3.333333333	1.1	1.999548665
11	1.617977528	1.8	1.090909091	1.996039121
12	1.618055556	1.87037037	1.083333333	1.996043893
13	1.618025751	1.920792079	1.076923077	1.996042682
14	1.618037135	1.953608247	1.071428571	1.996038309
15	1.618032787	1.920844327	1.066666667	1.996032261
16	1.618034448	1.925824176	1.0625	1.996025321
17	1.618033813	1.927960057	1.058823529	1.9960179
18	1.618034056	1.928227895	1.055555556	1.996038204
19	1.618033963	1.927283193	1.052631579	1.996031236
20	1.618033999	1.927526132	1.05	1.99603122
21	1.618033985	1.927590125	1.047619048	1.996031195
22	1.61803399	1.927576228	1.045454545	1.996031172
23	1.618033988	1.927551744	1.043478261	1.996031158
24	1.618033989	1.927561837	1.041666667	1.996031156
25	1.618033989	1.927563179	1.04	1.996031167
26	1.618033989	1.927562167	1.038461538	1.996031194
27	1.618033989	1.927561638	6.962962963	1.99603118
28	1.618033989	1.92756201	1.893617021	1.99603118
29	1.618033989	1.927562017	1.941011236	1.99603118
30	1.618033989	1.927561973	1.968162084	1.99603118
31	1.618033989	1.927561966	1.983088235	1.99603118
32	1.618033989	1.927561978	1.991101224	1.99603118
33	1.618033989	1.927561977	1.995344507	1.99603118
34	1.618033989	1.927561975	1.997573495	1.99603118
35	1.618033989	1.927561975	1.998738554	1.99603118
36	1.618033989	1.927561976	1.995605526	1.99603118
37	1.618033989	1.927561976	1.995830112	1.99603118
38	1.618033989	1.927561975	1.995944645	1.99603118
39	1.618033989	1.927561975	1.996001094	1.99603118
40	1.618033989	1.927561975	1.996026964	1.99603118
41	1.618033989	1.927561975	1.996036769	1.99603118
42	1.618033989	1.927561975	1.996038143	1.99603118
43	1.618033989	1.927561975	1.996035096	1.99603118
44	1.618033989	1.927561975	1.996029726	1.99603118
45	1.618033989	1.927561975	1.996030569	1.99603118
46	1.618033989	1.927561975	1.996030968	1.99603118
47	1.618033989	1.927561975	1.99603114	1.99603118
48	1.618033989	1.927561975	1.996031199	1.99603118
49	1.618033989	1.927561975	1.996031208	1.99603118

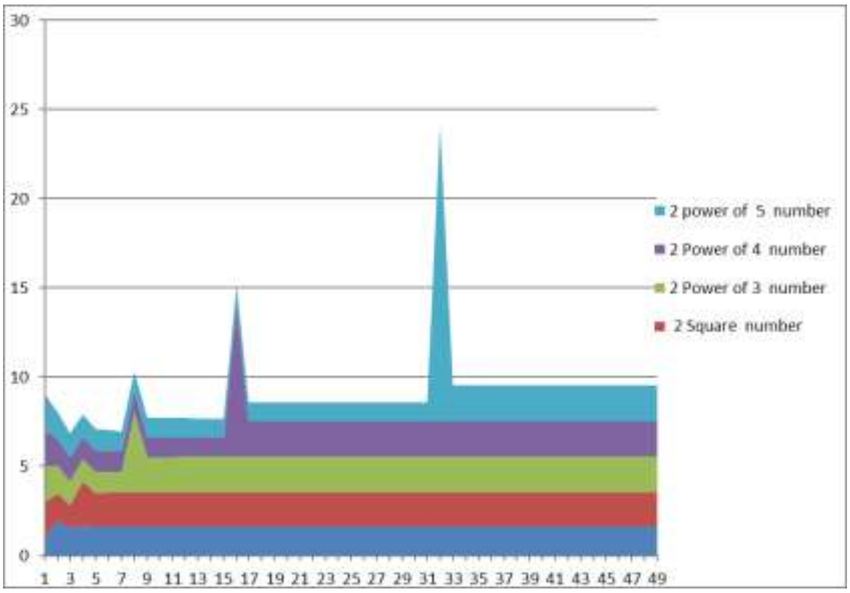
S. No	Conventional Fibonacci series (2 number sum)	4^2 Number sum series	4^3 Number sum series	…
1	1	1	64	
2	1	2	2080	
3	2	3	4159	
4	3	4	8316	
5	5	5	16629	
6	8	6	33254	
7	13	7	66503	
8	21	8	133000	
9	34	9	265993	
10	55	10	531978	
11	89	11	1063947	
12	144	12	2127884	
13	233	13	4255757	
14	377	14	8511502	
15	610	15	17022991	
16	987	16	34045968	
17	1597	58	68091921	
18	2584	103	136183826	
19	4181	192	272367635	
20	6765	369	544735252	
21	10946	722	1089470485	
22	17711	1386	2178940950	
23	28657	2669	4357881879	
24	46368	5146	8715763736	
25	75025	9923	17431527449	
26	121393	19124	34863054874	
27	196418	36862	69726109723	
28	317811	71055	139452219420	
29	514229	136964	278904438813	
30	832040	264005	557808877598	
31	1346269	508886	1115617755167	
32	2178309	980910	2231235510304	
33	3524578	1890765	4462471020577	
34	5702887	3644566	8924942041122	
35	9227465	7025127	17849884082211	
36	14930352	13541368	35699768164388	
37	24157817	26101826	71399536328741	
38	39088169	50312887	142799072657446	
39	63245986	96981208	285598145314855	
40	102334155	186937289	571196290629672	
41	165580141	360333210	1142392581259300	
42	267914296	694564594	2284785162518570	
43	433494437	1338816301	4569570325037100	
44	701408733	2580651394	9139140650074160	
45	1134903170	4974365499	18278281300148300	
46	1836311903	9588397788	36556562600296500	
47	2971215073	18482230982	73113125200592900	
48	4807526976	35625645663	146226250401186000	
49	7778742049	68670639932	292452500802372000	
50	12586269025	132366914365	584905001604743000	

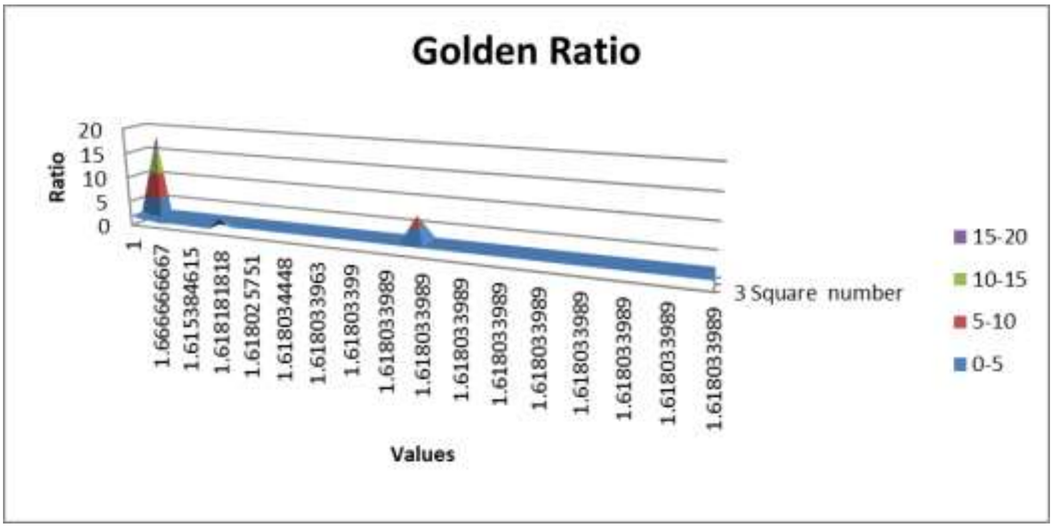
***Although only 4^3 number series are shown here, '4^n' number of such series is possible.*

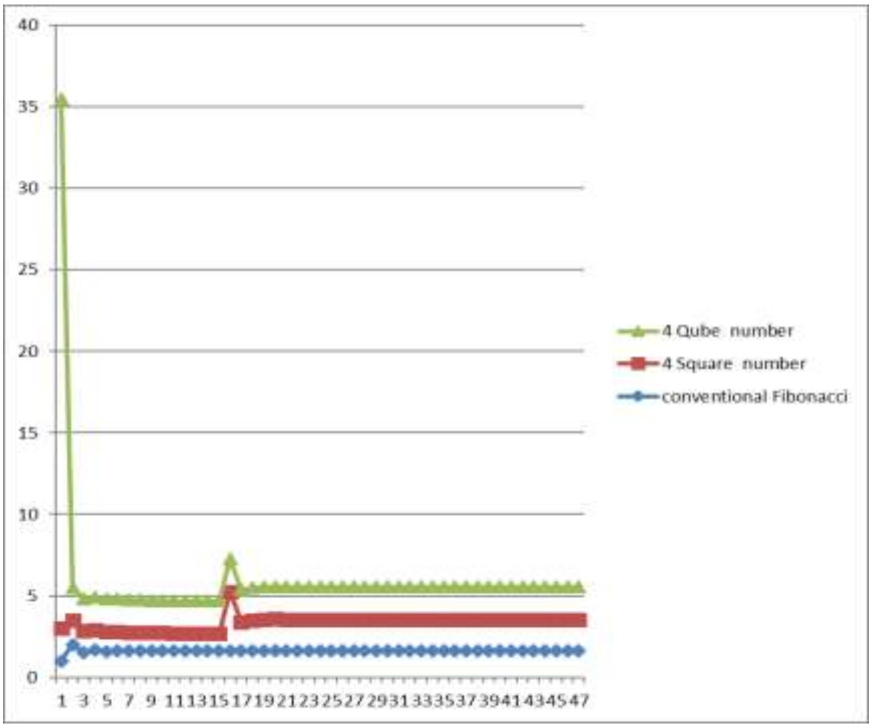
Now, if we calculate the ratio of any two numbers (such as the golden ratio for the Fibonacci series) for these '4^n' number sum series, and I denote it as $\Phi(4^n)$, it will tend to reach but never cross the '2' mark.

S. No	Conventional Fibonacci Golden Ratio [$\Phi(2)$]	Golden Ratio [$\Phi(4^2)$]	Golden Ratio [$\Phi(4^3)$]	**…**
1	1	2	32.5	**…**
2	2	1.5	1.999519231	**…**
3	1.5	1.333333333	1.999519115	**…**
4	1.666666667	1.25	1.99963925	**…**
5	1.6	1.2	1.999759456	**…**
6	1.625	1.166666667	1.999849642	**…**
7	1.615384615	1.142857143	1.999909779	**…**
8	1.619047619	1.125	1.999947368	**…**
9	1.617647059	1.111111111	1.999969924	**…**
10	1.618181818	1.1	1.999983082	**…**
11	1.617977528	1.090909091	1.999990601	**…**
12	1.618055556	1.083333333	1.999994831	**…**
13	1.618025751	1.076923077	1.99999718	**…**
14	1.618037135	1.071428571	1.999998473	**…**
15	1.618032787	1.066666667	1.999999178	**…**
16	1.618034448	3.625	1.999999559	**…**
17	1.618033813	1.775862069	1.999999765	**…**
18	1.618034056	1.86407767	1.999999875	**…**
19	1.618033963	1.921875	1.999999934	**…**
20	1.618033999	1.956639566	1.999999965	**…**
21	1.618033985	1.91966759	1.999999982	**…**
22	1.61803399	1.925685426	1.99999999	**…**
23	1.618033988	1.928062945	1.999999995	**…**
24	1.618033989	1.92829382	1.999999997	**…**
25	1.618033989	1.927239746	1.999999999	**…**
26	1.618033989	1.927525622	1.999999999	**…**
27	1.618033989	1.927594813	2	**…**
28	1.618033989	1.927577229	2	**…**
29	1.618033989	1.927550305	2	**…**
30	1.618033989	1.927561978	2	**…**
31	1.618033989	1.927563344	2	**…**
32	1.618033989	1.927562162	2	**…**
33	1.618033989	1.927561595	2	**…**
34	1.618033989	1.92756202	2	**…**
35	1.618033989	1.927562021	2	**…**
36	1.618033989	1.927561972	2	**…**
37	1.618033989	1.927561964	2	**…**
38	1.618033989	1.927561978	2	**…**
39	1.618033989	1.927561977	2	**…**
40	1.618033989	1.927561975	2	**…**
41	1.618033989	1.927561975	2	**…**
42	1.618033989	1.927561976	2	**…**
43	1.618033989	1.927561976	2	**…**
44	1.618033989	1.927561975	2	**…**
45	1.618033989	1.927561975	2	**…**
46	1.618033989	1.927561975	2	**…**
47	1.618033989	1.927561975	2	**…**
48	1.618033989	1.927561975	2	**…**
49	1.618033989	1.927561975	2	**…**

The plots of $\Phi(2^n), \Phi(3^n)$ and $\Phi(4^n)$ with n=1, 2, 3, 4,... will produce a seesaw plot, as shown below, with the line connecting the tips of the seesaw always being a straight line [5].


30
25
20
15
10
5
0
1 3 5 7 9 11 13 15 17 19 21 23 25 27 29 31 33 35 37 39 41 43 45 47 49
2 power of 5 number
2 Power of 4 number
2 Power of 3 number
2 Square number


Golden Ratio
Ratio
20
15
10
5
0
1
1.666666667
1.615384615
1.618181818
1.618025751
1.618034448
1.618033963
1.61803399
1.618033989
1.618033989
1.618033989
1.618033989
1.618033989
1.618033989
1.618033989
1.618033989
1.618033989
Values
3 Square number
15-20
10-15
5-10
0-5


40
35
30
25
20
15
10
5
0
1 3 5 7 9 11131517192123252729313335373941434547
4 Qube number
4 Square number
conventional Fibonacci

CONCLUSION:

Let us conclude with Leonardo-Da-words. Vinici's "Learn to see, and understand that everything is connected to everything else." While only a few examples are shown in this paper, the Fibonacci sequence and golden mean appear repeatedly, from the quantum level to the entire universe. The pervasiveness of these numbers is undoubtedly not a coincidence, and it may even point to a larger connection that has yet to be discovered. Any additional research into Phi and the Fibonacci sequence is welcomed and will contribute to a better understanding of beauty and spirituality in life. Finally, beautiful items in the world that are in golden ratio form are in golden ratio.

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