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On The System of Double Diophantine Equations

$x - y = u^2, \frac{x}{D} - y = v^2$

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Abstract:

The thrust of this paper is to obtain many non-zero distinct integer values of x such that $x - y = u^2, \frac{x}{D} - y = v^2$ where $D \geq 0$ & square-free and y is a known integer. A few numerical examples are given. The recurrence relation satisfied by the values of x is presented.

Keywords : System of double Diophantine equations, Diophantine problem

Introduction:

The theory of Diophantine equation is a treasure house in which the search for many hidden relations and properties form a treasure hunt. In fact, Diophantine problems dominated most of the celebrated unsolved mathematical problems. Certain diophantine problems come from physical problems or from immediate mathematical generalizations and others come from geometry in a variety of ways .Certain diophantine problems are neither trivial nor difficult to analyze. In this context, one may refer [1-18]. The above results motivated us to search for the integer solutions to some other choices of double diophantine equations.

In this paper, many non-zero distinct integer values of x such that $x - y = u^2, \frac{x}{D} - y = v^2$ where $D \geq 0$ & square-free and y is a known integer are obtained. A few numerical examples are given. The recurrence relation satisfied by the values of x is presented.

Method of analysis:

The system of equations to be solved is

$$x - y = u^2 \qquad (1)$$

$$\frac{x}{D} - y = v^2 \qquad (2)$$

where D is a non-zero integer > 1 and square-free

The elimination of x between (1) and (2) gives

$$u^2 = Dv^2 + y(D-1) \qquad (3)$$

Let $(u_\alpha^0, v_\alpha^0, y_\alpha)$ be an integral solution of (3) for given D and the corresponding value of x is

$$x_0 = (u_\alpha^0)^2 + y_\alpha$$

To obtain the other values of x, we proceed as follows:

Let $(\tilde{u}_n, \tilde{v}_n)$ be the general solution of the Pellian

$$u^2 = Dv^2 + 1 \qquad (4)$$

given by

$$\tilde{u}_n + \sqrt{D}\,\tilde{v}_n = (\tilde{u}_0 + \sqrt{D}\,\tilde{v}_0)^{n+1}, \; n = 0,1,2,....$$

in which $(\widetilde{u}_0, \widetilde{v}_0)$ is the initial solution of (4).

Applying the lemma of Brahmagupta between the solutions (u_α^0, v_α^0) and $(\widetilde{u}_n, \widetilde{v}_n)$, the sequence of values of u and v satisfying (3) are given by

$$v_n = u_\alpha^0 \widetilde{v}_n + v_\alpha^0 \widetilde{u}_n$$

$$v_n = \frac{1}{2\sqrt{D}} \left\{ \left(u_\alpha^0 + \sqrt{D} v_\alpha^0\right)\left(\widetilde{u}_0 + \sqrt{D}\, \widetilde{v}_0\right)^{n+1} - \left(u_\alpha^0 - \sqrt{D} v_\alpha^0\right)\left(\widetilde{u}_0 - \sqrt{D}\, \widetilde{v}_0\right)^{n+1} \right\}$$

$$u_n = u_\alpha^0 \widetilde{u}_n + D v_\alpha^0 \widetilde{v}_n$$

$$u_n = \frac{1}{2} \left\{ \left(u_\alpha^0 + \sqrt{D} v_\alpha^0\right)\left(\widetilde{u}_0 + \sqrt{D}\, \widetilde{v}_0\right)^{n+1} + \left(u_\alpha^0 - \sqrt{D} v_\alpha^0\right)\left(\widetilde{u}_0 - \sqrt{D}\, \widetilde{v}_0\right)^{n+1} \right\}$$

Substituting the values of u_n and y_α in (1), the sequence of values of x are obtained as

$$x_{n+1} = u_n^2 + y_\alpha \;,\; n = 0,1,2,....$$

The values of x satisfy the following recurrence relation

$$\sqrt{x_{n+2} - y_\alpha} - 2\widetilde{u}_0 \sqrt{x_{n+1} - y_\alpha} + \sqrt{x_n - y_\alpha} = 0$$

To analyze the nature of the solutions one has to go in for particular values of D and y.
A few illustrations are given below:

ILLUSTRATION I:

Let $D = 3$ .Then

: $y_\alpha = 2\alpha^2 + 4\alpha - 4 \;,\; u_\alpha^0 = 2\alpha + 2 \;,\; v_\alpha^0 = 2 \;,\; \widetilde{u}_0 = 2 \;,\; \widetilde{v}_0 = 1$

Thus the values of x satisfying (1) and (2) are given by

$$x_0 = 6\alpha^2 + 12\alpha$$

$$x_{n+1} = \frac{1}{4}\left\{\left((2\alpha+2) + 2\sqrt{3}\right)\left(2+\sqrt{3}\right)^{n+1} + \left((2\alpha+2) - 2\sqrt{3}\right)\left(2-\sqrt{3}\right)^{n+1}\right\}^2 + \left(2\alpha^2 + 4\alpha - 4\right)$$

Some numerical examples are:

Table: 1

n	Values of x when		
	$\alpha = 1$ $y_1 = 2$	$\alpha = 2$ $y_2 = 12$	$\alpha = 3$ $y_3 = 26$
-1	**18**	**48**	**90**
0	**198**	**336**	**510**
1	**2706**	**4368**	**6426**

: $y_\alpha = 2\alpha^2 + 2\alpha - 1 \;,\; u_\alpha^0 = 2\alpha + 1 \;,\; v_\alpha^0 = 1 \;,\; \widetilde{u}_0 = 2 \;,\; \widetilde{v}_0 = 1$

The values of x satisfying (1) and (2) are given by

$$x_0 = 6\alpha^2 + 6\alpha$$

$$x_{n+1} = \frac{1}{4}\left\{\left((2\alpha+1) + \sqrt{3}\right)\left(2+\sqrt{3}\right)^{n+1} + \left((2\alpha+1) - \sqrt{3}\right)\left(2-\sqrt{3}\right)^{n+1}\right\}^2 + \left(2\alpha^2 + 2\alpha - 1\right)$$

A few values of x are presented below:

Table: 2

n	Values of x when		
	$\alpha=1$ $y_1=3$	$\alpha=2$ $y_2=11$	$\alpha=3$ $y_3=23$
-1	**12**	**36**	**72**
0	**84**	**180**	**312**
1	**1092**	**2220**	**3744**

The above solutions in (I) and (II) satisfying the following recurrence relation

$$\sqrt{x_{n+2}-y_\alpha}-4\sqrt{x_{n+1}-y_\alpha}+\sqrt{x_n-y_\alpha}=0$$

ILLUSTRATION 2:

Let $D=5$.

: $y_\alpha=\alpha^2$, $u_\alpha^0=3\alpha$, $v_\alpha^0=\alpha$, $\tilde{u}_0=9$, $\tilde{v}_0=4$

A few values of x are presented below:

Table: 3

n	Values of x when		
	$\alpha=1$ $y_1=1$	$\alpha=2$ $y_2=4$	$\alpha=3$ $y_3=9$
-1	**10**	**40**	**90**
0	**2210**	**8840**	**19890**
1	**710650**	**2842600**	**6395850**

(IV) : $y_\alpha=\alpha^2+3\alpha+1$, $u_\alpha^0=3\alpha+2$, $v_\alpha^0=\alpha$, $\tilde{u}_0=9$, $\tilde{v}_0=4$

Table: 4

n	Values of x when		
	$\alpha=1$ $y_1=5$	$\alpha=2$ $y_2=11$	$\alpha=3$ $y_3=19$
-1	**30**	**75**	**140**
0	**4230**	**12555**	**25300**
1	**1357230**	**4032075**	**8128220**

The above patterns in (III) and (IV) satisfy the recurrence relation

$$\sqrt{x_{n+2}-y_\alpha}-18\sqrt{x_{n+1}-y_\alpha}+\sqrt{x_n-y_\alpha}=0$$

ILLUSTRATION 3:

Let $D=6$.Then

: $y_\alpha=5\alpha^2+2\alpha-1$, $u_\alpha^0=5\alpha+1$, $v_\alpha^0=1$, $\tilde{u}_0=5$, $\tilde{v}_0=2$

Table: 5

n	Values of x when		
	$\alpha=1$ $y_1=6$	$\alpha=2$ $y_2=23$	$\alpha=3$ $y_3=50$
-1	**42**	**144**	**306**
0	**1770**	**4512**	**8514**
1	**171402**	**434304**	**817266**

(VI) : $y_\alpha=\frac{1}{4}\left[5\alpha^4-10\alpha^3+21\alpha^2-16\alpha+8\right]$, $u_\alpha^0=\frac{1}{2}\left[5\alpha^2-5\alpha+8\right]$,
$v_\alpha^0=1,\ \tilde{u}_0=5,\ \tilde{v}_0=2$

Table: 6

n	Values of x when		
	$\alpha=1$ $y_1=2$	$\alpha=2$ $y_2=15$	$\alpha=3$ $y_3=71$
-1	**18**	**96**	**432**
0	**1026**	**3264**	**11520**
1	**99858**	**314736**	**1104672**

Patterns in (V) and (VI) satisfy the following recurrence relation:

$$\sqrt{x_{n+2}-y_\alpha}-10\sqrt{x_{n+1}-y_\alpha}+\sqrt{x_n-y_\alpha}=0$$

As a special case of the system of equations (1) and (2), we consider the pair of equations given by

$$x-y^2=u^2 \tag{5}$$

$$\frac{x}{D}-y^2=v^2 \tag{6}$$

for its non-trivial integral solutions.

Eliminating x between the equations (5) and (6), we have

$$u^2=Dv^2+(D-1)y^2 \tag{7}$$

To analyze the nature of the solutions, one has to go in for finding solutions of equation (7) for partial values of D.

For example, when $D=2$, the equation (7) becomes

$$u^2=2v^2+y^2 \tag{8}$$

for which the solution is well-known. Thus, using the standard characterizations of equation (8) the patterns of solutions satisfying the system of equations (5) and (6) are given as follows:

Pattern I:

Here,

$$x=8r^4+2s^4\ ,\ y=2r^2-s^2$$

From the above solutions, we observe the following:

Table: 7

r	s	x	y
Even	**Even**	**Even**	**Even**
Odd	**Odd**	**Even**	**Odd**
Even	**Odd**	**Even**	**Odd**
Odd	**Even**	**Even**	**Even**

For s odd, the solutions are co-primes

Pattern II:

The equations are

$$x = 2r^4 + 8s^4 \ , \ y = r^2 - 2s^2$$

The nature of the solutions for different choices of r and s are seen as below:

Table: 8

r	s	x	y
Even	**Even**	**Even**	**Even**
Odd	**Odd**	**Even**	**Odd**
Even	**Odd**	**Even**	**Even**
Odd	**Even**	**Even**	**Odd**

When r is odd, the solutions are co-primes.

Another method of finding solutions of the equation (7) is given below:

On introducing the linear transformations

$$v = X \mp (D-1), \ y = X \pm D \tag{9}$$

The equation (7) becomes

$$u^2 = (2D-1)X^2 + D(D-1)(2D-1) \tag{10}$$

which, on solving, gives u and X. Using the values of X in (9), the corresponding values of y and v are obtained. Knowing the values of u and y, the values of x are obtained from (5). [We may also obtain x by using the values of y and v in (6)].

The nature of solutions of (10) can be analyzed only for the particular values of D, as the general form of integral solutions of (10) is not possible. For the sake of clear understanding, we classify the solutions into two parts A and B namely, choose D such that (A): $2D-1$ is square-free and (B): $2D-1$ is a perfect square. The process of obtaining integral solutions is explained through the following examples:

Part (A):

(i) Let $D = 4$. The equation (10) simplifies to

$$u^2 = 7X^2 + 84$$

whose solutions are

$$X_n = \frac{1}{2^{n-1}}\left[\left(3+\sqrt{7}\right)^{2n} + \left(3-\sqrt{7}\right)^{2n}\right] + \frac{\sqrt{7}}{2^n}\left[\left(3+\sqrt{7}\right)^{2n} - \left(3-\sqrt{7}\right)^{2n}\right]$$

$$u_n = \frac{\sqrt{7}}{2^n}\left\{\left(2+\sqrt{7}\right)\left(3+\sqrt{7}\right)^{2n} - \left(2-\sqrt{7}\right)\left(3-\sqrt{7}\right)^{2n}\right\},\ n = 1,2,3....$$

Substituting the above values in (9), the values of x and y satisfying the given system of equations are given by

$$y_n = \frac{1}{2^{n-1}}\left[\left(3+\sqrt{7}\right)^{2n} + \left(3-\sqrt{7}\right)^{2n}\right] + \frac{\sqrt{7}}{2^n}\left[\left(3+\sqrt{7}\right)^{2n} - \left(3-\sqrt{7}\right)^{2n}\right] - 4$$

$$x_n = y_n^2 + u_n^2$$

where $n = 1,2,3....$

(ii) Setting $D = 8$ in (10), we get

$$y_n = X_n - D = \left\{\left(4+\sqrt{15}\right)^n\left(1+\sqrt{15}\right) + \left(4-\sqrt{15}\right)^n\left(1-\sqrt{15}\right)\right\} - 8$$

$$x_n = y_n^2 + u_n^2 = \left(X_n - D\right)^2 + 15\alpha_n^2$$

where $\alpha_n = \frac{1}{\sqrt{15}}\left\{\left(4+\sqrt{15}\right)^n\left(\sqrt{15}+1\right) + \left(4-\sqrt{15}\right)^n\left(\sqrt{15}-1\right)\right\},\ n = 1,2,3,.....$

Part (B):

For the choice $2D - 1 =$ a perfect square, say α^2, it is possible to obtain a general formula for the integral solutions of the system of equations under consideration as follows:

Replacing $(D-1)$ by $\alpha^2 - D$ in equation (7), we have

$$u^2 - \alpha^2 y^2 = D\left(v^2 - y^2\right)$$

which is written as

$$\frac{u - \alpha y}{v - y} = \frac{D(v+y)}{u + \alpha y} = \frac{q}{p} \qquad (11)$$

where p and q are non-zero distinct integers.

The equation (11) is equivalent to the system of homogeneous equations

$$pu - qv + (q - \alpha p)y = 0$$

$$qu - Dpv + (\alpha q - pD)y = 0$$

Applying the method of cross multiplication, we have

$$\left.\begin{aligned} u &= t\left(2pqD - \alpha q^2 - \alpha p^2 D\right) \\ v &= t\left(q^2 - 2pq\alpha + p^2 D\right) \\ y &= t\left(q^2 - p^2 D\right) \end{aligned}\right\} \qquad (12)$$

Substituting the values of u and y given by (12) in (5), we obtain the corresponding value of x is given by

$$x = t^2\left[\left(1+\alpha^2\right)\left(q^4 + p^4D^2\right) - 2p^2q^2\left(1 - \alpha^2 - 2D\right) - 4pqD\left(q^2 + \alpha p^2 D\right)\right]$$

Examples:

Table: 9

p	q	D	α	u	v	t	x	y
1	**2**	**5**	**3**	**-7**	**-3**	**1**	**50**	**-1**
3	**2**	**13**	**5**	**-449**	**61**	**1**	**214370**	**-113**

GENERATION OF SOLUTIONS:

Knowing a solution of equation (7), a method to generate a sequence of solutions is presented below:

Let $v_1 = v_0 + mp, y_1 = y_0 + mq, u_1 = u_0 + mr$ (13)

where p, q, r and m are non-zero distinct integers, be a second solution of equation (7)

Using (13) in (7), we obtain

$$m = \frac{2(ru_0 - Dpv_0 - (D-1)qy_0)}{Dp^2 + (D-1)q^2 - r^2}$$

Thus the second solution of equation (7) in matrix form is given by

$$\begin{pmatrix} v_1 \\ y_1 \\ u_1 \end{pmatrix} = M \begin{pmatrix} v_0 \\ y_0 \\ u_0 \end{pmatrix}$$

where M is the matrix represented by

$$M = \begin{pmatrix} (D-1)q^2 - r^2 - Dp^2 & -2pq(D-1) & 2pr \\ -2pqD & Dp^2 - r^2 - (D-1)q^2 & 2qr \\ -2prD & -2qr(D-1) & Dp^2 + (D-1)q^2 + r^2 \end{pmatrix}$$

where $Dp^2 + (D-1)q^2 \neq r^2$

Repeating the above process, the sequence of integral solutions of equation (7) are obtained from the following relations

$$\begin{pmatrix} v_{2n} \\ y_{2n} \\ u_{2n} \end{pmatrix} = \lambda^n \begin{pmatrix} v_0 \\ y_0 \\ u_0 \end{pmatrix} \text{ and}$$

$$\begin{pmatrix} v_{2n-1} \\ y_{2n-1} \\ u_{2n-1} \end{pmatrix} = \lambda^{n-1} M \begin{pmatrix} v_0 \\ y_0 \\ u_0 \end{pmatrix} \quad , \; n = 1, 2, 3, ... \qquad (14)$$

in which $\lambda = ((D-1)q^2 - r^2 - Dp^2)^2 + 4p^2D[(D-1)q^2 - r^2]$

Therefore, knowing the values of y, u and v, one can obtain the corresponding values of x from (5) or (6).

From (14), we observe that

$$\lambda\begin{pmatrix} v_{2n-1} \\ y_{2n-1} \\ u_{2n-1} \end{pmatrix} = M\begin{pmatrix} v_{2n} \\ y_{2n} \\ u_{2n} \end{pmatrix}$$

Conclusion:

In this paper, an attempt has been made to obtain many non-zero distinct integer values of x such that $x - y = u^2, \frac{x}{D} - y = v^2$ where $D \geq 0$ & square-free and y is a known integer. One may search for other choices of system of double equations and determine their solutions

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