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Observations on Homogeneous Bi-quadratic Equation with Five unknowns

$$(x^3 + y^3)(x - y) = 19(z^2 - w^2)P^2$$

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Abstract

This paper concerns with the problem of determining non-trivial integral solutions of the homogeneous bi-quadratic equation with five unknowns given by $(x^3 + y^3)(x - y) = 19(z^2 - w^2)P^2$.We obtain infinitely many non-zero integer solutions of the equation by introducing the linear transformations.

Keywords: Bi-quadratic equation with five unknowns, integral solutions, homogeneous bi-quadratic, Linear Transformations.

Introduction

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular, bi-quadratic diophantine equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity[1-5]. In this context, one may refer [6-13] for various problems on the bi-quadratic diophantine equations with five variables. However, often we come across homogeneous bi-quadratic equations and as such one may require its integral solution in its most general form. It is towards this end, this paper concerns with the problem of determining non-trivial integral solutions of the homogeneous equation with five unknowns given by $(x^3 + y^3)(x - y) = 19(z^2 - w^2)P^2$

Method of analysis:

The homogeneous bi-quadratic diophantine equation with five unknowns under consideration is

$$(x^3 + y^3)(x - y) = 19(z^2 - w^2)P^2 \tag{1}$$

Introduction of the linear transformations

$$x = u + v, y = u - v, z = uv + 1, w = uv - 1 \tag{2}$$

in (1) leads to

$$u^2 + 3v^2 = 19P^2 \tag{3}$$

which can be solved through different methods.

Pattern.1

Assume

$$P = a^2 + 3b^2 \tag{4}$$

Write 19 as

$$19 = (4 + i\sqrt{3})(4 - i\sqrt{3}) \tag{5}$$

Using (4) and (5) in (3) and employing the method of factorization ,we get

$$(u+\sqrt{3}iv)(u-\sqrt{3}iv)=(4+i\sqrt{3})(4-i\sqrt{3})(a+i\sqrt{3}b)^2(a-i\sqrt{3}b)^2$$

On equating the positive and negative factors, we have ,

$$(u+i\sqrt{3}v)=(4+i\sqrt{3})(a+i\sqrt{3}b)^2$$

$$(u-\sqrt{3}iv)=(4-i\sqrt{3})(a-i\sqrt{3}b)^2$$

On equating real and imaginary parts, we get,

$$u=u(a,b)=4a^2-12b^2-6ab$$

$$v=v(a,b)=a^2-3b^2+8ab$$

Substituting the values of u and v in (2) ,we get the values of x, y, z and w as

$$\left.\begin{aligned} x&=x(a,b)=5a^2-15b^2+2ab\\ y&=y(a,b)=3a^2-9b^2-14ab\\ z&=z(a,b)=4a^4-92a^2b^2+26a^3b-78ab^3+36b^4+1\\ w&=w(a,b)=4a^4-92a^2b^2+26a^3b-78ab^3+36b^4-1\end{aligned}\right\} \quad (6)$$

Thus (4) and (6) represent non-zero distinct integer solutions of (1).

Note: It is worth to mention here that 19 may be represented as follows:

$$19=\frac{(7+3i\sqrt{3})(7-3i\sqrt{3})}{2^2} \quad (7)$$

Following the analysis as that of Pattern I one may obtained different set of integer solution to (1) as,

$$x=x(A,B)=20A^2-60B^2-8AB$$

$$y=y(A,B)=8A^2-24B^2-64AB$$

$$z=z(A,B)=84A^2+756B^4-1512A^2B^2+176A^3B-528AB^3+1$$

$$w=w(A,B)=84A^2+756B^4-1512A^2B^2+176A^3B-528AB^3-1$$

$$P=P(A,B)=4A^2+12B^2$$

Pattern.II

Rewrite the equation (3) as

$$19P^2-3v^2=u^2*1 \quad (8)$$

Write 1 as

$$1=\frac{(\sqrt{19}+\sqrt{3})(\sqrt{19}-\sqrt{3})}{16} \quad (9)$$

Assume

$$u=19a^2-3b^2=(\sqrt{19}a+\sqrt{3}b)(\sqrt{19}a-\sqrt{3}b) \quad (10)$$

Using (10) and (9) in (8) and using the method of factorization ,we get

$$(\sqrt{19}P+\sqrt{3}v)(\sqrt{19}P-\sqrt{3}v)=(\sqrt{19}a+\sqrt{3}b)^2(\sqrt{19}a-\sqrt{3}b)^2\left(\frac{(\sqrt{19}+\sqrt{3})(\sqrt{19}-\sqrt{3})}{4^2}\right)$$

Equating the positive and negative parts, we have

$$(\sqrt{19}P+\sqrt{3}v)=(\sqrt{19}a+\sqrt{3}b)^2\left(\frac{(\sqrt{19}+\sqrt{3})}{4}\right) \quad (11)$$

$$(\sqrt{19}P-\sqrt{3}v)=(\sqrt{19}a-\sqrt{3}b)^2\left(\frac{(\sqrt{19}-\sqrt{3})}{4}\right) \quad (12)$$

Equating the coefficients of $\sqrt{19}$ and $\sqrt{3}$ in the equation (11),we get

$$P=\frac{19a^2+3b^2+6ab}{4} \quad (13)$$

$$v=\frac{19a^2+3b^2+38ab}{4} \quad (14)$$

Substituting the values of u, v and assuming $a=2A, b=2B$,the non-zero distinct integral solutions of (1) are obtained as

$$x=x(A,B)=95A^2-9B^2+38AB$$

$$y=y(A,B)=57A^2-15B^2-38AB$$

$$z=z(A,B)=1444A^4-36B^4+2888A^3B-456AB^3+1$$

$$w=w(A,B)=1444A^4-36B^4+2888A^3B-456AB^3-1$$

$$P=P(A,B)=19A^2+3B^2+6AB$$

Note: It is worth to mention here that in (8),1 may also be represented as follows:

$$1=(2\sqrt{19}+5\sqrt{3})(2\sqrt{19}-5\sqrt{3})$$

Following the same procedure as above, one may get different set of solutions as below.

$$x(a,b)=114a^2+12b^2+76ab$$

$$y(a,b)==-76a^2-18b^2-76ab$$

$$z(a,b)=1805a^4-45b^4+1444a^3b-228ab^3+1$$

$$w(a,b)=1805a^4-45b^4+1444a^3b-228ab^3-1$$

$$P(a,b)=38a^2+6b^2+30ab$$

Pattern.III

Rewrite the equation (3) as

$$u^2+3v^2=19*P^2*1 \quad (15)$$

Write 1 as

$$1=\frac{(1+i\sqrt{3})(1-i\sqrt{3})}{2^2} \qquad (16)$$

Using (4),(5) and (16) in (15) and employing the method of factorization, we get,

$$(u+i\sqrt{3}v)(u-i\sqrt{3}v)=(4+i\sqrt{3})(4-i\sqrt{3})(a+ib\sqrt{3})^2(a-ib\sqrt{3})^2\frac{(1+i\sqrt{3}((1-i\sqrt{3})}{2^2}$$

Equating the positive parts and negative parts, we get, $(u+i\sqrt{3}v)=(4+i\sqrt{3})\dfrac{(1+i\sqrt{3})}{2}(a+ib\sqrt{3})^2$ (17)

$$(u-i\sqrt{3}v)=(4-i\sqrt{3})\frac{(1-i\sqrt{3})}{2}(a-ib\sqrt{3})^2 \qquad (18)$$

Equating real and imaginary parts of (17), we get,

$$u=\frac{1}{2}\left[a^2-3b^2-30ab\right]$$

$$v=\frac{1}{2}\left[5a^2-15b^2+2ab\right]$$

Substituting the values of u,v and assuming $a=2A,b=2B$,the non-zero distinct integral solutions of (1) are obtained as

$$x=x(A,B)=12A^2-36B^2-56AB$$

$$y=y(A,B)=-8A^2+24B^2-64AB$$

$$z=z(A,B)=20A^4+180B^4-360A^2B^2-592A^3B+1776AB^3+1$$
$$w=w(A,B)=20A^4+180B^4-360A^2B^2-592A^3B+1776AB^3-1$$

$$P=P(A,B)=4A^2+12B^2$$

Note: It is worth to mention here that 1 may be represented as follows:

$$1=\frac{(1+4i\sqrt{3})(1-4i\sqrt{3})}{7^2} \qquad (19)$$

Using (4),(5) and (19) in (15) ,and employing the same procedure as above and replacing a by $7A$ and b by $7B$,we get the solution as

$$x=x(A,B)=63A^2-189B^2-826AB$$

$$y=y(A,B)=-175A^2+525B^2-602AB$$

$$z=z(A,B)=49[-136A^4+2448A^2B^2-1224B^4-1606A^3B+4818AB^3]+1$$

$$w=w(A,B)=49[-136A^4+2448A^2B^2-1224B^4-1606A^3B+4818AB^3]-1$$

$$P=P(A,B)=49A^2+147B^2$$

Pattern.IV

Introducing the linear transformations

$$P=X+3T,u=4U,v=X+19T \qquad (20)$$

In the equation (3), it leads to

$$X^2 = U^2 + 57T^2 \tag{21}$$

which is satisfied by

$$\left.\begin{array}{l} U = 57r^2 - s^2 \\ X = 57r^2 + s^2 \\ T = 2rs \end{array}\right\} \tag{22}$$

Using (20),(22) and (2),we obtained the non-zero distinct integer solution of (1) as

$$x(r,s) = 285r^2 - 3s^2 + 38rs$$

$$y(r,s) = 171r^2 - 5s^2 - 38rs$$

$$z(r,s) = 12996r^4 - 4s^2 + 8664r^3 s - 152rs^3 + 1$$

$$w(r,s) = 12996r^4 - 4s^2 + 8664r^3 s - 152rs^3 - 1$$

$$P(r,s) = 57r^2 + s^2 + 6rs$$

Note: It is to be noted that (21) may be represented as

$$(X+U)(X-U) = 57T^2$$

Then we have the system of double equation as shown in the following table.

TABLE

System	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$X+U$	$57T^2$	$19T^2$	$3T^2$	T^2	57	19	3	$57T$	$19T$	T
$X-U$	1	3	19	57	T^2	$3T^2$	$19T^2$	T	$3T$	$57T$

Employing the procedure as above in each of the above representations, the corresponding solutions to (1) are represented below.

Solution for system:1

$$x(k) = 570k^2 + 608k + 160$$

$$y(k) = 342k^2 + 304k + 64$$

$$z(k) = 51984k^4 + 121296k^3 + 103968k^2 + 38912k + 5377$$

$$w(k) = 51984k^4 + 121296k^3 + 103968k^2 + 38912k + 5375$$

$$P(k) = 114k^2 + 120k + 32$$

Solution for system:2

$$x(k) = 190k^2 + 228k + 62$$

$$y(k) = 144k^2 + 76k + 2$$

$$z(k) = 5776k^4 + 17328k^3 + 17328k^2 + 6992k + 961$$

\ $w(k)=5776k^4+17328k^3+17328k^2+6992k+959$

$P(k)=38k^2+44k+14$

Solution for system:3

$x(k)=30k^2+68k-2$

$y(k)=18k^2-20k-62$

$z(k)=144k^4+1200k^3+15384k^2-688k-959$

$w(k)=144k^4+1200k^3+15384k^2-688k-961$

$P(k)=6k^2+12k+14$

Solution for system:4

$x(k)=10k^2+48k-64$

$y(k)=6k^2-32k-160$

$z(k)=16k^4+336k^3+480k^2-4096k-5375$

$w(k)=16k^4+336k^3+480k^2-4096k-5377$

$P(k)=2k^2+8k+32$

Solution for system:5

$x(k)=-6k^2+32k+160$

$y(k)=-10k^2-48k+64$

$z(k)=-16k^4-336k^3-480k^2+4096k+5377$

$w(k)=-16k^4-336k^3-480k^2+4096k+5375$

$P(k)=2k^2+8k+32$

Solution for system:6

$x(k)=-18k^2+20k+62$

$y(k)=-30k^2-68k+2$

$z(k)=-144k^4-1200k^3-1584k^2+688k+961$

$w(k)=-144k^4-1200k^3-1584k^2+688k+959$

$P(k)=6k^2+12k+14$

Solution for system:7

$x(k)=-144k^2-76k-2$

$$y(k) = -190k^2 - 228k - 62$$

$$z(k) = -5776k^4 - 17328k^3 - 17328k^2 - 6992k - 959$$

$$w(k) = -5776k^4 - 17328k^3 - 17328k^2 - 6992k - 961$$

$$P(k) = 38k^2 + 44k + 14$$

Solution for system:8

$$x(k) = 160k$$

$$y(k) = 64k$$

$$z(k) = 5376k^2 + 1$$

$$w(k) = 5376k^2 - 1$$

$$P(k) = 32k$$

Solution for system:9

$$x(k) = 62k$$

$$y(k) = 2k$$

$$z(k) = 960k^2 + 1$$

$$w(k) = 960k^2 - 1$$

$$P(k) = 14k$$

Solution for system:10

$$x(k) = -64k$$

$$y(k) = -160k$$

$$z(k) = -5376k^2 + 1$$

$$w(k) = -5376k^2 - 1$$

$$P(k) = 32k$$

Pattern. V

Write (3) as

$$u^2 - 16P^2 = 3(P^2 - v^2)$$

$$\Rightarrow (u + 4P)(u - 4P) = 3(P + v)(P - v) \qquad (23)$$

which can be expressed in the form of ratio as

$$\frac{(u + 4P)}{3(P + v)} = \frac{(P - v)}{(u - 4P)} = \frac{A}{B}, \quad (A \neq B \neq 0)$$

This is equivalent to following system of equations

$$Bu-3Av+P(4B-3A)=0$$

$$Au+Bv-P(B+4A)=0$$

Solving these two equations using cross multiplication method, we get the values of u,v and P

as

$$u=12A^2-4B^2+6AB$$

$$v=-3A^2+B^2+8AB$$

$$P=3A^2+B^2$$

Substituting the above values of u and v in (2) we obtained the nonzero distinct integer solution of (1) as

$$x(A,B)=9A^2-3B^2+14AB$$

$$y(A,B)=15A^2-5B^2-2AB$$

$$z(A,B)=-36A^4-4B^4+72A^2B^2+78A^3B-26AB^3+1$$

$$w(A,B)=-36A^4-4B^4+72A^2B^2+78A^3B-26AB^3-1$$

$$P(A,B)=3A^2+B^2$$

Note: It is observed that (23) may also be represented as in the following cases:

$$i)\ \frac{(\mathrm{u}-4\mathrm{P})}{(\mathrm{P}-\mathrm{v})}=\frac{3(P+v)}{u+4P}=\frac{A}{B}\ (\mathrm{A}\neq\mathrm{B}\neq 0)$$

$$ii)\ \frac{\mathrm{u}+4\mathrm{P}}{\mathrm{P}-\mathrm{v}}=\frac{3(P+v)}{u-4P}=\frac{A}{B}\ (\mathrm{A}\neq\mathrm{B}\neq 0)$$

$$iii)\ \frac{(\mathrm{u}-4\mathrm{P})}{3(\mathrm{P}-\mathrm{v})}=\frac{P+v}{u+4P}=\frac{A}{B}\ (\mathrm{A}\neq\mathrm{B}\neq 0)$$

Employing the procedure as above in each of the above representations, the corresponding distinct non-zero integer solutions to (1) are presented below.

Solution for case.(i)

$$x(A,B)=-3A^2+9B^2+14AB$$

$$y(A,B)=-5A^2+15B^2-2AB$$

$$z(A,B)=-4A^4-36B^4+72A^2B^2-26A^3B+78AB^3+1$$

$$w(A,B)=-4A^4-36B^4+72A^2B^2-26A^3B+78AB^3-1$$

$$P(A,B)=A^2+3B^2$$

Solution for case.(ii)

$$x(A,B)=5A^2-15B^2-2AB$$

$$y(A,B)=3A^2-9B^2+14AB$$

$$z(A,B) = 4A^4 + 36B^4 - 72A^2B^2 - 26A^3B + 78AB^3 + 1$$

$$w(A,B) = 4A^4 + 36B^4 - 72A^2B^2 - 26A^3B + 78AB^3 - 1$$

$$P(A,B) = A^2 + 3B^2$$

Solution for case.(iii)

$$x(A,B) = -9A^2 + 3B^2 + 14AB$$

$$y(A,B) = -15A^2 + 5B^2 - 2AB$$

$$z(A,B) = -36A^4 - 4B^4 - 24A^2B^2 - 114A^3B + 38AB^3 + 1$$

$$w(A,B) = -36A^4 - 4B^4 - 24A^2B^2 - 114A^3B + 38AB^3 - 1$$

$$P(A,B) = 3A^2 + B^2$$

Conclusion:

In this paper, a search is made for obtaining different choice of integer solutions to homogeneous bi-quadratic diophantine equation with five unknowns $(x^3 + y^3)(x - y) = 19(z^2 - w^2)P^2$. To conclude, as bi-quadratic equations are rich in variety, the researchers may search for integer solutions to the other types of bi-quadratic equations with variables greater than or equal to five.

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