



## Graphical Comparison on Numerical Solutions of Initial Value Problem by Using Euler's Method, Modified Euler's Method and Runge-Kutta Method

<sup>1</sup>Muhammad Imran, <sup>2</sup>Murat Cancan, <sup>1</sup>Shahzad Rashid, <sup>1</sup>Yasir Ali, <sup>1</sup>Najma Imran,

<sup>1</sup> Department of Mathematics, Concordia College Kasur Campus, Punjab, Pakistan

<sup>2</sup> Faculty of Education, Yuzuncu Yil University, van, Turkey

DOI: <https://doi.org/10.55248/gengpi.2022.3.10.62>

### ABSTRACT

In this paper, discussion is made on the numerical solutions of ordinary differential equations of first order along with initial value problems by using Euler's Method, Modified Euler's Method and RK - 4 Methods. Solutions and graphs of some numerical examples have been obtained with the help of MATLAB program as well as we determined the exact analytical solutions. In order to validate the accuracy, we compared exact solutions with the numerical approximate solutions by using different step sizes.

**Keywords**—Euler's Method, Modified Euler's Method, Runge-Kutta Method

### I. INTRODUCTION

An equation containing the derivatives, ordinary derivatives or partial derivatives is called a differential equation shortly (DE). If an equation consists of ordinary derivatives only, then it is known as an ordinary differential equation shortly (ODE). Similarly, if an equation consists of partial derivatives only, then this is termed as a partial differential equation shortly (PDE). An order of a DE is the order of highest order derivative that appears in it. A first order DE is an equation which involves the dependent variable  $y$  and its first derivative. The solution of a DE is to determine a function  $y(x)$  which satisfies the given DE over an interval  $a < x < b$ . Initial conditions shortly (ICs) are the solution values or its derivatives at certain points. The initial value problem shortly (IVP) is a DE which consists of a suitable set of initially defined values.

In most real life situations, DEs are generally utilized within the disciplines of applied engineering and biological sciences to model the mathematical problems. In order to solve the mathematical problems which have been deduced in the form of differential equations, numerical methods are usually employed where it becomes very difficult situation or even impossible situation to solve a differential equation analytically. It has been seen that merely a finite set of DEs have exact solutions. Many exact methods exist to find the solutions of DEs. Still there exists a big collection of DEs, where we cannot find the solution in a 'closed form' by employing familiar exact methods. There, we must have to employ the numerical methods to find the numerical solutions of DEs under the suggested initial points.

In this work, we present three standard approximate methods, namely, Euler's Method, Modified Euler's Method and RK - 4 Method for solving IVP for ODEs of first order.

Euler's Method is formulated by Leonhard Euler, in 1768. It is also known as the tangent line method [27]. This is a very simpler one-step method. This is the first and foremost approximate method to solve IVP. It is applied to study the concepts in the advanced methods. This is the elementary explicitly defined method for the approximate solution of ODEs. It is very essential for studying because it makes the analysis of error very easiest to comprehend [17]. This method consists of three forms namely: Forward Euler's Method, Improved Euler's Method and Backward Euler's Method [9].

The curve for studying the Modified Euler's Method in an interval  $(x_0, x_1)$  where  $x_1 = x_0 + h$  is rounded by the straight line passing through the point  $(x_0, y_0)$  with slope  $f(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0))$ , which is the slope of the midpoint having  $x$ -coordinate is the arithmetic mean of  $x_0$  and  $x_1$  i.e.,  $x_0 + \frac{h}{2}$  [21].

RK - Method described by two German mathematicians Carl Runge and Martin Wilhelm Kutta in 1895 and 1905 respectively [8]. One of the most vastly known family members of Runge-Kutta is commonly mentioned as "RK - 4". It is also known as the "classical RK - Method" or in a simple manner "the RK - Method" [12]. The method is famous by its orders according to the notion that it is agreed with the solution of Taylor's series till terms of  $h^r$ , where the exponent 'r' denotes the order of a method. Moreover, this method does not require already calculation of higher differentiations

of  $y(x)$  like in Taylor's Method. The RK – 4 Method is broadly employed for finding approximate solutions of ODEs along with given initial conditions [18]. The 1<sup>st</sup> - order RK – Method is called Euler's Method. The Second order RK - Method is identical to the Modified Euler's or Heun's Method. The RK - Method of order 4 is called classical Runge-Kutta Method [8,22]. For further texts and details on these subjects see [6,11,12,19,20].

Stability concept and errors are explained in details in [7] and [25]. An approximate method is defined to be convergent method when the necessary and sufficient

conditions, that is, stability and consistency are satisfied. Stability concept belongs to the growing or decaying of an error when the numerical calculation is making progresses. Consistency deals with reducing the step size, the truncation error terms could be made to approach 0 [22].

In order to find the numerical solution of ODEs, often two kinds of errors are encountered, namely, round off errors and truncation errors. Round off errors come into being when computer can only display the numbers using a fixed and limited number of significant digits. Therefore, these numbers can be or cannot be displayed with exactness in the computer storage. The discrimination entered by this deficiency is termed as round off error. Truncation errors come into being when we make truncation from the infinite sum in order to approximate it into the finite sum [23]. Refining the mesh size for making less the truncation error frequently makes the round off error to grow. Exactness of the solution would rely upon to what level tiny step size is used. When the approximate solution goes to the analytical solution as the step size goes to 0, then the approximate method is said to be convergent [17].

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## II. PRELIMINARIES

### 2.1 Differential Equation

An equation that describes the rate of change of a one or more than one depending variable, w. r. t a single or more than one independent variable, is defined to be a differential equation, shortly we say DE. For example,

$$\frac{du}{dx} = cu + x^2 \quad (2.1)$$

$$\frac{d^2u}{dx^2} - x \frac{du}{dx} + u = 0 \quad (2.2)$$

$$\frac{\partial u}{\partial t} = 6u \frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3} \quad (2.3)$$

$$\frac{dy}{dx} = ax - bxy \quad (2.4)$$

$$r \frac{dr}{d\theta} = \sin \theta + \cos \theta \quad (2.5)$$

Equations (2.1), (2.2), (2.3), (2.4) and (2.5) are examples of DEs.

### 2.2 Kinds of Differential Equation

DEs may be classified according to the types of unknown functions appearing in the given equation.

Ordinary DE

Partial DE

### 2.3 Ordinary DE

An equation involving merely ordinary derivatives of a single or more than one depending variable, with respect to one independent variable, is called an ODE. For example,

$$\frac{dy}{dx} = 3xy \quad (2.9)$$

$$(4x + y) dx + xdy = 0 \quad (2.10)$$

$$\frac{d^2y}{dx^2} + 10 \frac{dy}{dx} + 9y = 0 \quad (2.11)$$

$$\frac{dy}{dx} = ycscx + e^x \quad (2.12)$$

Equations (2.9), (2.10), (2.11) and (2.12) are Ordinary DEs.

#### 2.3.1 Order of ODE

The highest order derivative that appears in a DE is defined to be an order of ODE. For example,

$$\frac{d^3y}{dx^3} - \frac{dy}{dx} + 2y = 0 \quad (2.13)$$

$$\frac{dy}{dx} + y \sin x = 5 \quad (2.14)$$

$$\frac{d^4y}{dx^4} - 2 \left(\frac{dy}{dx}\right)^5 + 3y = 4x \quad (2.15)$$

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = \frac{d^2y}{dx^2} \quad (2.16)$$

Equations (2.13), (2.14), (2.15) and (2.16) have orders 3, 1, 4 and 2 respectively.

### 2.3.2 Degree of ODE

The largest power of the highest order derivative which occurs in a DE is defined to be a degree of a DE if we express the depending variable and its rate of change in a shape independent from radical symbols and quotients. For example,

$$\frac{d^2y}{dx^2} + 3 \frac{dy}{dx} + 2y = 0 \quad (2.17)$$

$$\left(\frac{d^3y}{dx^3}\right)^3 + 3 \frac{d^2y}{dx^2} + \frac{dy}{dx} + 6y = 0 \quad (2.18)$$

$$y = x \frac{dy}{dx} + \sqrt{1 + \left(\frac{dy}{dx}\right)^4} \quad (2.19)$$

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = \frac{d^2y}{dx^2} \quad (2.20)$$

Equations (2.17), (2.18), (2.19) and (2.20) have degrees 1, 3, 4 and 2 respectively.

### 2.4 Partial DE (PDE)

The equation involving only partial derivatives of a single or more than one depending variable, w.r.t two or more than two independent variables, is called a partial DE, shortly we say PDE. For example,

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 8u \quad (2.55)$$

$$\frac{\partial^2 u}{\partial t^2} = 4 \frac{\partial^2 u}{\partial x^2} \quad (2.56)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad (2.57)$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} + \left(\frac{\partial^3 u}{\partial x^3}\right)^2 \quad (2.58)$$

Equations (2.55), (2.56), (2.57) and (2.58) are examples of PDEs.

#### 2.4.1 Order of PDE

The highest order partial derivative which appears in a PDE is defined as an order. For example,

$$\frac{\partial u}{\partial x} = z + x \frac{\partial u}{\partial y} \quad (2.59)$$

$$\frac{\partial^2 u}{\partial t^2} = 4 \frac{\partial^2 u}{\partial x^2} \quad (2.60)$$

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} + \left(\frac{\partial u}{\partial x}\right)^2 \quad (2.61)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad (2.62)$$

Equations (2.59), (2.60), (2.61) and (2.62) are of orders 1, 2, 3 and 2 respectively.

#### 2.4.2 Degree of PDE

The largest power of the highest order partial derivative that appears in a PDE is defined as a degree if we express the depending variable and its partial derivatives all being in a shape independent from radicals and fractions. For example,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (2.63)$$

$$\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2}\right)^{\frac{1}{3}} \quad (2.64)$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} + \left(\frac{\partial^3 u}{\partial x^3}\right)^2 \quad (2.65)$$

$$\frac{\partial u}{\partial t} = c \left(\frac{\partial^2 u}{\partial x^2}\right)^3 \quad (2.66)$$

Equations (2.63), (2.64), (2.65) and (2.66) have degrees 1, 1, 2 and 3 respectively.

## 2.5 Solution forms of DE

### 2.5.1 General Solution (GS)

A solution which is obtained after the integration of the Differential Equation defined as a General Solution (GS). The  $n$ th-order Ordinary DE has a General Solution containing  $n$  arbitrary constants which comes into being after  $n$  times integrating the given DE.

### 2.5.2 Particular Solution (PS)

A solution which is acquired after providing specific values in place of unknown constants that occurs in a General Solution is known as a Particular Solution (PS).

### 2.5.3 Singular Solution

The Solution that may not be expressed by a General Solution is known as a Singular Solution.

### 2.5.4 Trivial Solution

When the solution of a homogeneous DE is happened along with zero value of a depending variable, then this Solution is known as a Trivial Solution.

### 2.5.5 Nontrivial Solution

The solution of the given problem which exists differently than the Trivial Solution is said to be a Non-Trivial Solution.

## 2.6 Auxiliary Conditions

The GS of a DE containing the same number of unknown constants as is the order of an equation. ODE which describes the physical system commonly occurs with the solutions more than one. Occasionally it is required to determine the solutions of DE with respect to additional conditions, called auxiliary conditions. For this, different kinds of conditions are discussed. (i) Initial Conditions and (ii) Boundary Conditions

### 2.6.1 Initial Conditions

Specific conditions are given in order to determine the solution of a DE. When such conditions are given at the same initial point of the independent variable with  $y = y_0$  over  $x = x_0$ , it is also to write as  $y(x_0) = y_0$  and  $(\frac{dy}{dx})_0 = y'(x_0)$  over  $x = x_0$ , and  $x_0 \in ] a, b [$ , these conditions are characterized as initial conditions, shortly ICs and  $x_0$  is known as the initial value.

#### 2.6.1.1 Initial Value Problem (IVP)

When all the required values are provided simply at the same initial point, then this mathematical problem is known as the Initial Value Problem, shortly we say IVP. It is also called Starting Problem or Marching Problem, because the solution is advanced in steps. An IVP consists of two parts. (i) The Differential Equation  $y' = \frac{dy}{dx} = f(x, y)$ , which gives the relationship between  $y(x)$  and  $y'(x)$  and (ii) Initial Condition  $y(x_0) = y_0$ .

The solutions of Initial-Value Problems are easily obtained by using Direct Methods. For example,

$$y' = x - y^2; y(0) = 1 \quad (2.72)$$

$$y' = 3x^2 + y; y(0) = 1 \quad (2.73)$$

$$\frac{dx}{dt} = 2x + y, \frac{dy}{dt} = x - 2y; x(1) = 2, y(1) = 3 \quad (2.74)$$

$$y'' + y = 0; y(0) = 0, y'(0) = 1 \quad (2.75)$$

Equations (2.72), (2.73), (2.74) and (2.75) are examples of IVPs.

### 2.6.2 Boundary Conditions

When the conditions are given at two different values of the independent variable with  $y = y_0$  at  $x = a$  and  $y = y_1$  at  $x = b$  written as  $y(a) = y_0$  and  $y(b) = y_1$ , where  $x \in [a, b]$  are known as the boundary conditions, shortly BCs.

#### 2.6.2.1 Boundary Value Problem (BVP)

When the conditions are given at two or more than two distinct points, then this mathematical problem is defined as BVP. Sometimes it is also called

Jury Problem. The Boundary Value Problem (BVP) consists of two parts. (i) The simplest differential equation  $y'' = f(x, y, y')$  and (ii) Boundary Conditions  $y(a) = y_0$  and  $y(b) = y_1$

The solutions of BVPs are, in general, determined iteratively. For example,

$$y'' + y' + 2y = 0; y(1) = 1, y(2) = 2 \quad (2.76)$$

$$y'' - y + x + \cosh x = 0; y(0) = 0, y(1) = 1 \quad (2.77)$$

Equations (2.76) and (2.77) are examples of BVPs.

## 2.7 Methods to solve ODEs

Many methods are developed to solve ODEs, for instance, (i) Analytical methods (ii) Graphical methods, and (iii) Numerical methods

### 2.7.1 Analytical Methods

If a problem can be solved analytically, it is commonly assumed being very much exactitude solution. Most of the ODEs encountered in practice either may not have a possible solution by exact methods or these have sufficiently tiresome exact solutions due to the involvement of a big number of integrals. Therefore, in order to find the numerical solutions we would ponder over the other methods.

### 2.7.2 Graphical Methods

The Graphical Methods may give very useful insight into the nature of the solutions of ODEs, but here we also face many adverse shortcomings, a few of them are as follows. (i) Draughtsman's Technique leads to limit the accuracy (ii) Subjective judgment are done and (iii) It is very difficult to compute the error. Owing to absent any basic methods, we find to find the numerical solutions of ODEs.

We ponder over three numerical approximate methods for obtaining the numerical solution of the 1<sup>st</sup>-order ODEs along with initial value problem (IVP) is of the shape

$$y' = f(x, y(x)), y(x_0) = y_0, x_0 \leq x \leq x_n \quad (5.1)$$

Where  $y' = \frac{dy}{dx}$  and  $f(x, y(x))$  is a given function and  $y(x)$  is a solution of an equation (5.1). Here we shall try to find out the solution of the equation (5.1) on a given finite range  $x_0 \leq x \leq x_n$ , along with given initial point  $x_0$ . A continuous values of the solution  $y(x)$  shall not be found possibly but approximate solution to  $y$  shall be produced at different values, which we called them mesh points, in the range  $x_0 \leq x \leq x_n$ . We shall use equation (5.1) to get approximate solutions for the values of a solution which corresponds to different chosen values of the solution  $x = x_n = x_0 + nh, n = 0, 1, 2, \dots$ . The number  $h$  is said to be a step length. The approximate solution to an eqn. (5.1) is known by a collection of points  $\{(x_0, y_n); n = 0, 1, 2, 3, \dots, n\}$  and every point  $(x_0, y_n)$  shows the approximation with the corresponding points  $(x_n, y(x_n))$  on the solution curve.

- **The generalized form of Euler's Method is given by**

$$y_{n+1} = y_n + hf(x_n, y_n); n = 0, 1, 2, 3, \dots$$

- **The generalized form of Modified Euler's Method is given by**

$$y_{n+1} = y_n + hf\left\{x_n + \frac{h}{2}, y_n + \frac{h}{2}f(x_n, y_n)\right\}; n = 0, 1, 2, 3, \dots$$

- **The generalized form of RK-4 Method is given by**

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4); n = 0, 1, 2, 3, \dots$$

where,

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_n + h, y_n + k_3)$$

The correctness of a solution will rely on how tiny we make the step length  $h$ . The approximate method is assumed to be convergent method whether  $\lim_{h \rightarrow 0} \max_{1 \leq n \leq N} |y(x_n) - y_n| = 0$ , where  $y(x_n)$  represents the numerical solution and  $y_n$  represents the analytical solution. The maximum error is defined by  $E_R = \lim_{h \rightarrow 0} \max_{1 \leq n \leq N} |y(x_n) - y_n|$ .

Here we solve ordinary differential equations of 1<sup>st</sup>-order along with initial conditions by applying EM, MEM and RK-4 Methods. Firstly we find the numerical solutions by using EM, MEM and RK-4 Methods. Secondly we make comparison between numerical solutions and exact solutions for various step lengths like  $h = 0.1, 0.05, 0.025, 0.0125$ . Thirdly we discuss error of these proposed methods.

We consider three numerical problems of 1<sup>st</sup>-order ODEs in order to show that which approximate method converges swiftly to exactitude solution. The approximated solutions and graphs have been determined by utilizing MATLAB software for three selected methods at various step lengths. Approximate findings and errors are calculated and results are compared.

### III. MAIN RESULTS

We look at the (IVP)  $y'(x) = 3x^2 + y$ ;  $y(0) = 1$  in the range  $0 \leq x \leq 1$ . Then the analytical solution of the problem is known as  $y(x) = -3(x^2 + 2x + 2) + 7e^x$ . The numerical solutions and utmost errors are acquired and demonstrated as tabular form in 5.1. (1) – (4) and approximate solutions are represented by graphs in fig - 5.1. (1) – (12).

**Table (Example 3.1)** (1) Approximate solutions and utmost errors of step length  $h = 0.1$  (2) Approximate solutions and utmost errors of step length  $h = 0.05$  (3) Approximate solutions and utmost errors of step length  $h = 0.025$  (4) Approximate solutions and utmost errors of step length  $h = 0.0125$ .

#### 3.1. (1) Approximate solutions and utmost errors of step length $h = 0.1$

	Eulers Method	Errors	Modified Eulers Method	Errors	RK-4 Method	Errors	Exact Solution
$x_n$	$y(x_n)$	$E_R$	$y(x_n)$	$E_R$	$y(x_n)$	$E_R$	$y_n$
0	1.0000000000000000	0	1.0000000000000000	0	1.0000000000000000	0	1.0000000000000000
0.1	1.1000000000000000	6.19643e-03	1.1057500000000000	4.46427e-04	1.1061964583333333	3.18038e-08	1.106196426529534
0.2	1.2130000000000000	1.68193e-02	1.2287537500000000	1.06556e-03	1.229819311686632	4.56544e-08	1.229819307121188
0.3	1.3463000000000000	3.27117e-02	1.3771228937500000	1.88876e-03	1.379011558546141	9.44859e-08	1.379011653032022
0.4	1.5079300000000000	5.48429e-02	1.5598207975937500	2.95209e-03	1.562772603334738	2.80154e-07	1.562772883488893
0.5	1.7067230000000000	8.43259e-02	1.786751981341094	4.29691e-03	1.791048325337955	5.69563e-07	1.791048894900897
0.6	1.9523953000000000	1.22436e-01	2.068860939381909	5.97066e-03	2.074830620254019	9.82480e-07	2.074831602733562
0.7	2.2556348300000000	1.70634e-01	2.418241338017010	8.02761e-03	2.426267410611651	1.54168e-06	2.426268952293338
0.8	2.6281983130000000	2.30588e-01	2.848256678508796	1.05298e-02	2.858784226075187	2.27337e-06	2.858786499447271
0.9	3.0830181443000000	3.04204e-01	3.373673629752219	1.35481e-02	3.387218570451703	3.20765e-06	3.387221778098649
1.0	3.6343199587300000	3.93653e-01	4.010809360876203	1.71634e-02	4.027968420188250	4.37903e-06	4.027972799213316

#### 3.1. (2) Approximate solutions and utmost errors of step length $h = 0.05$

	Eulers Method	Errors	Modified Eulers Method	Errors	RK-4 Method	Errors	Exact Solution
$x_n$	$y(x_n)$	$E_R$	$y(x_n)$	$E_R$	$y(x_n)$	$E_R$	$y_n$
0	1.0000000000000000	0	1.0000000000000000	0	1.0000000000000000	0	1.0000000000000000
0.1	1.1028750000000000	3.32143e-03	1.1060782421875000	1.18184e-04	1.106196427944138	1.41460e-09	1.106196426529534
0.2	1.2208696875000000	8.94962e-03	1.229537859707214	2.81447e-04	1.229819306034392	1.08680e-09	1.229819307121188
0.3	1.3616838304687500	1.73278e-02	1.378513690549341	4.97962e-04	1.379011644688472	8.34355e-09	1.379011653032022
0.4	1.533806423091797	2.89665e-02	1.561995760258482	7.77123e-04	1.562772862160655	2.13282e-08	1.562772883488893
0.5	1.746596581458706	4.44523e-02	1.789919184861530	1.12971e-03	1.791048853735481	4.11654e-08	1.791048894900897
0.6	2.010372731058224	6.44589e-02	2.073263524981325	1.56808e-03	2.074831533580734	6.91528e-08	2.074831602733562
0.7	2.336510935991691	8.97580e-02	2.424162584706744	2.10637e-03	2.426268845507941	1.06785e-07	2.426268952293338
0.8	2.737553306930840	1.21233e-01	2.856025753240579	2.76075e-03	2.858786343664949	1.55782e-07	2.858786499447271
0.9	3.227327520891251	1.59894e-01	3.383672102777735	3.54968e-03	3.387221559980879	2.18118e-07	3.387221778098649
1.0	3.821078591782604	2.06894e-01	4.023478583632405	4.49422e-03	4.027972503157892	2.96055e-07	4.027972799213316

#### 3.1. (3) Approximate solutions and utmost errors of step length $h = 0.025$

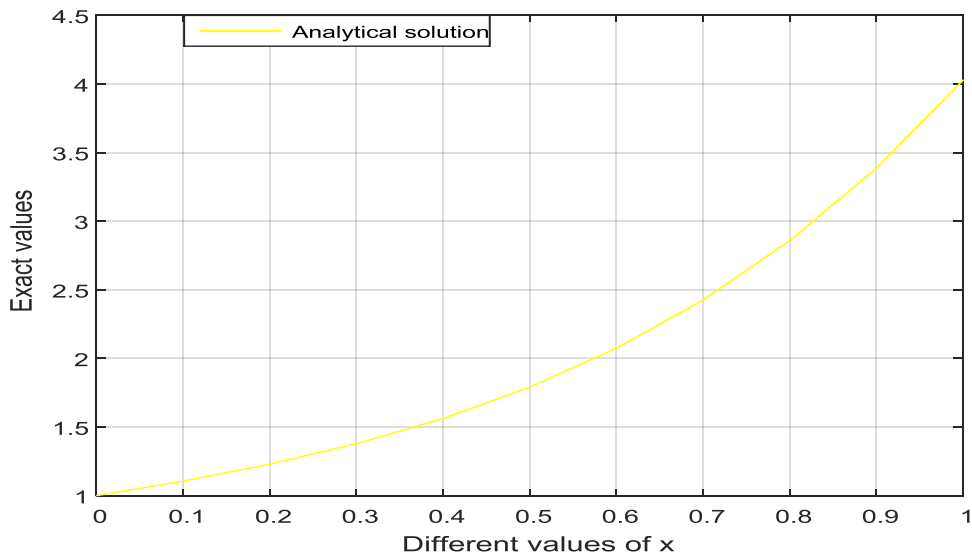
	Eulers Method	Errors	Modified Eulers Method	Errors	RK-4 Method	Errors	Exact Solution
$x_n$	$y(x_n)$	$E_R$	$y(x_n)$	$E_R$	$y(x_n)$	$E_R$	$y_n$
0	1.0000000000000000	0	1.0000000000000000	0	1.0000000000000000	0	1.0000000000000000
0.1	1.104476201171875	1.72023e-03	1.106166022432862	3.04041e-05	1.106196426598834	6.92999e-11	1.106196426529534
0.2	1.225200499882671	4.61881e-03	1.229746983770742	7.23234e-05	1.229819307007687	1.13501e-10	1.229819307121188
0.3	1.370088431542561	8.92322e-03	1.378883809542723	1.27843e-04	1.379011652429818	6.02203e-10	1.379011653032022
0.4	1.547877266173491	1.48956e-02	1.562573521909982	1.99362e-04	1.562772882029768	1.45913e-09	1.562772883488893

0.5	1.768211315054559	2.28376e-02	1.790759266461980	2.89628e-04	1.791048892144587	2.75631e-09	1.791048894900897
0.6	2.041736093296831	3.30955e-02	2.074429806196315	4.01797e-04	2.074831598156672	4.57689e-09	2.074831602733562
0.7	2.380202257707244	4.60667e-02	2.425729478247638	5.39474e-04	2.426268945276727	7.01661e-09	2.426268952293338
0.8	2.796580334742609	6.22062e-02	2.858079713624361	7.06786e-04	2.858786489261716	1.01856e-08	2.858786499447271
0.9	3.305187358704145	8.20344e-02	3.386313335914658	9.08442e-04	3.387221763888509	1.42101e-08	3.387221778098649
1.0	3.921826656609056	1.06146e-01	4.026822982793303	1.14982e-03	4.027972779978040	1.92353e-08	4.027972799213316

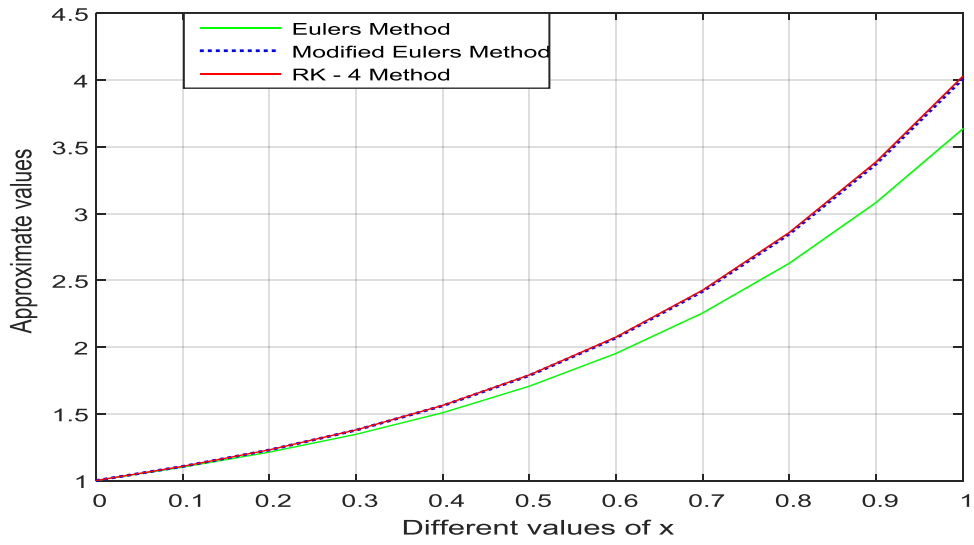
**3.1. (4) Approximate solutions and utmost errors of step length  $h = 0.0125$**

	Eulers Method	Errors	Modified Eulers Method	Errors	RK-4 Method	Errors	Exact Solution
$x_n$	$y(x_n)$	$E_R$	$y(x_n)$	$E_R$	$y(x_n)$	$E_R$	$y_n$
0	1.000000000000000	0	1.000000000000000	0	1.000000000000000	0	1.000000000000000
0.1	1.105320937064189	8.75489e-04	1.106188715971968	7.71056e-06	1.106196426533248	3.71370e-12	1.106196426529534
0.2	1.227472691959278	2.34662e-03	1.229800976059274	1.83311e-05	1.229819307112627	8.56115e-12	1.229819307121188
0.3	1.374483017290997	4.52864e-03	1.378979264727283	3.23883e-05	1.379011652991788	4.02338e-11	1.379011653032022
0.4	1.555218454236098	7.55443e-03	1.562722395792159	5.04877e-05	1.562772883393647	9.52469e-11	1.562772883488893
0.5	1.779471974289856	1.15769e-02	1.790975570923297	7.33240e-05	1.791048894722741	1.78155e-10	1.791048894900897
0.6	2.058059778358067	1.67718e-02	2.074729909424362	1.01693e-04	2.074831602439338	2.94222e-10	2.074831602733562
0.7	2.402928210008769	2.33407e-02	2.426132445368997	1.36507e-04	2.426268951843826	4.49510e-10	2.426268952293338
0.8	2.827271839672799	3.15147e-02	2.858607692922662	1.78807e-04	2.858786498796297	6.50972e-10	2.858786499447271
0.9	3.345663887002076	4.15579e-02	3.386991996454356	2.29782e-04	3.387221777192028	9.06610e-10	3.387221778098649
1.0	3.974200270551608	5.37725e-02	4.027682009989886	2.90789e-04	4.027972797987710	1.22560e-09	4.027972799213316

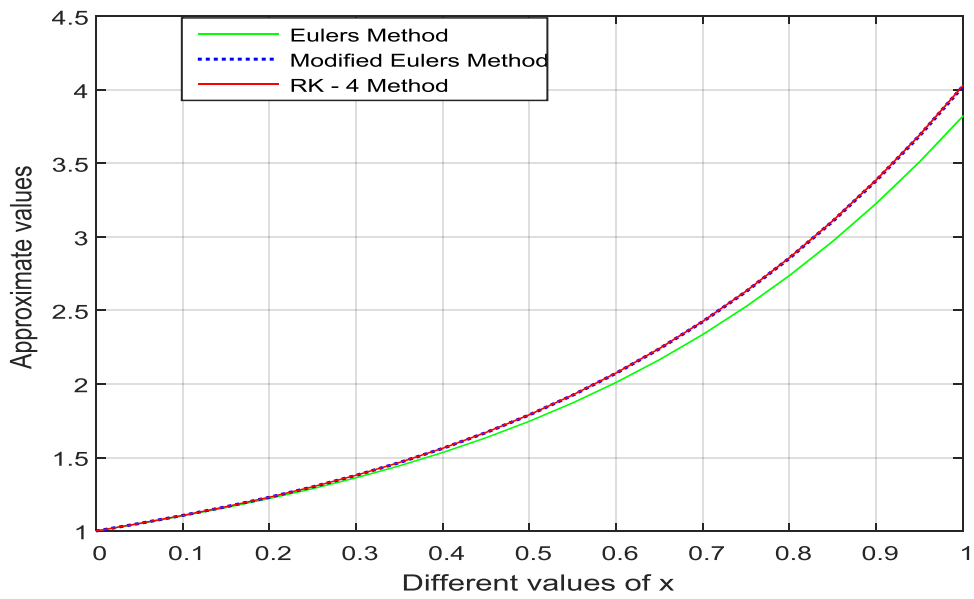
**MATLAB Graphs of Example (3.1)**



**Fig - 3.1.1 Analytical solution**

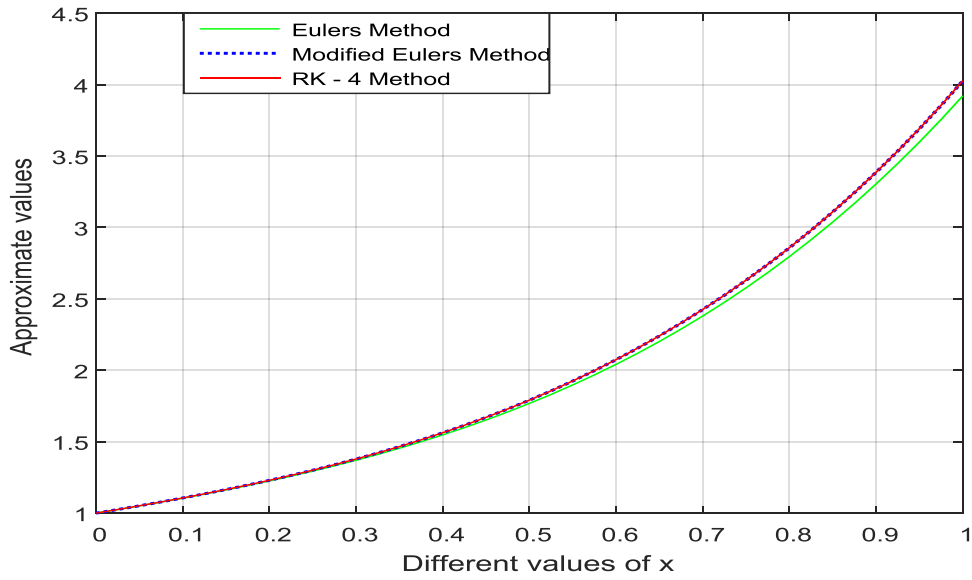


**Fig - 3.1.2** Approximate solutions of step length  $h = 0.1$

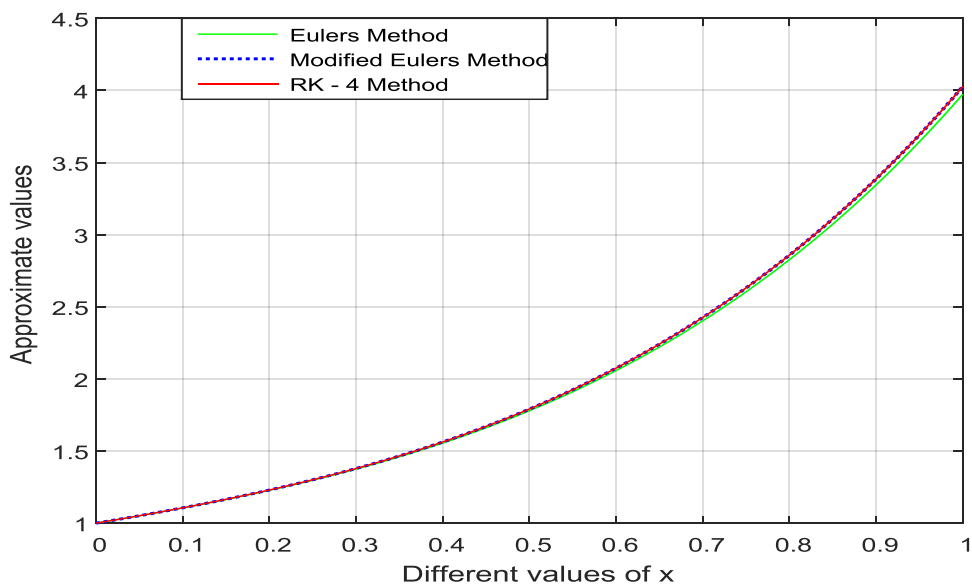


**Fig - 3.1.3** Approximate solutions of step length  $h = 0.05$

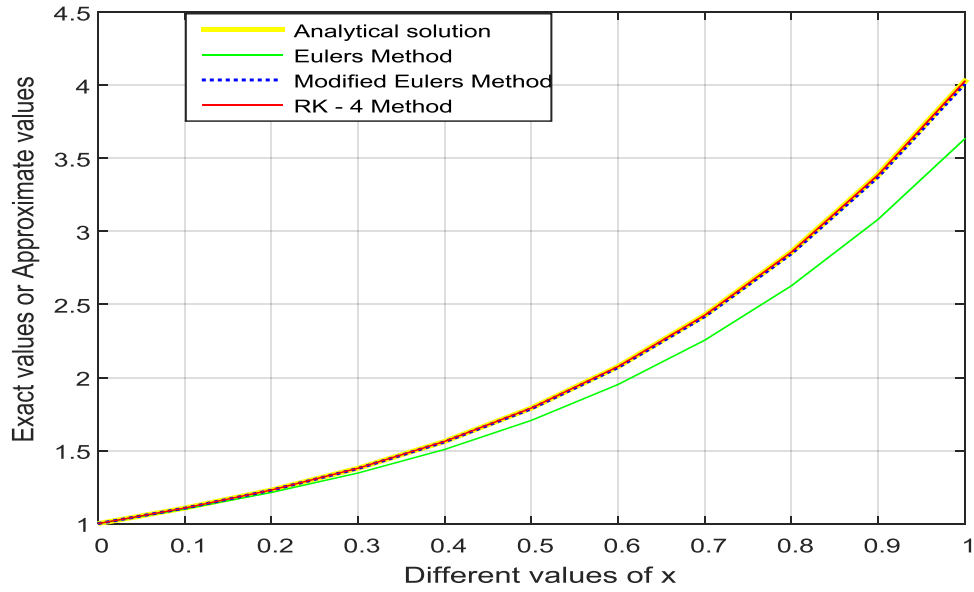




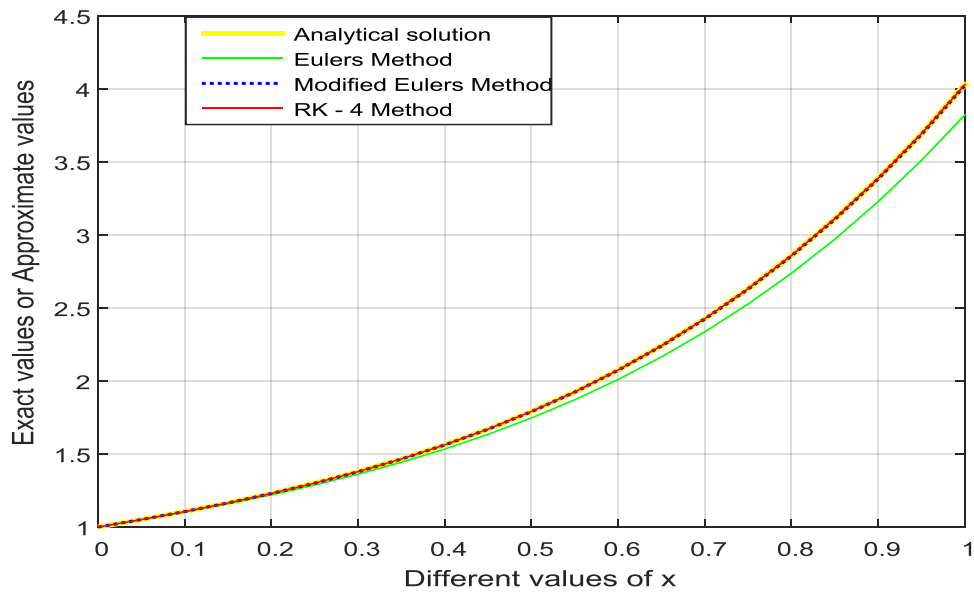
**Fig - 3.1.4** Approximate solutions of step length  $h = 0.025$



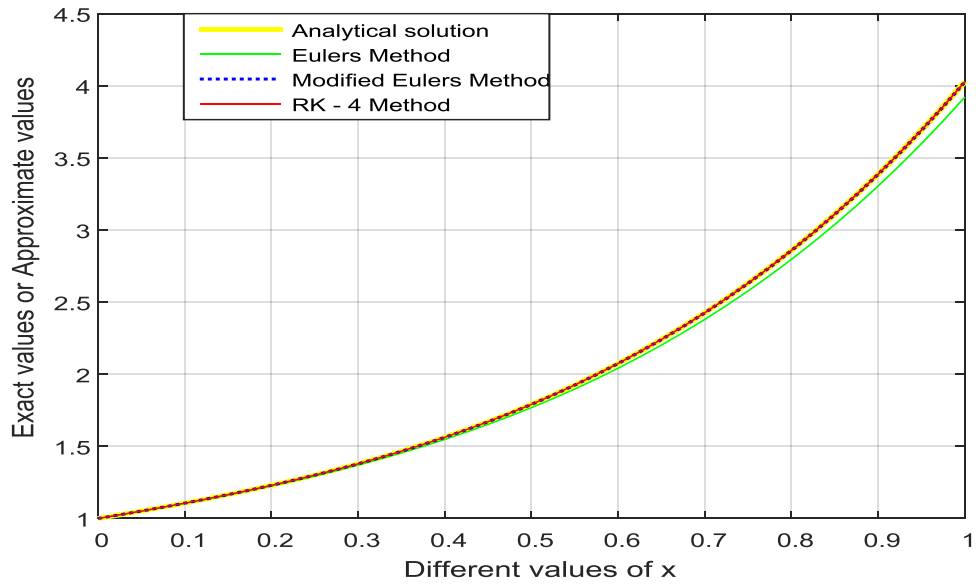
**Fig - 3.1.5** Approximate solutions of step length  $h = 0.0125$



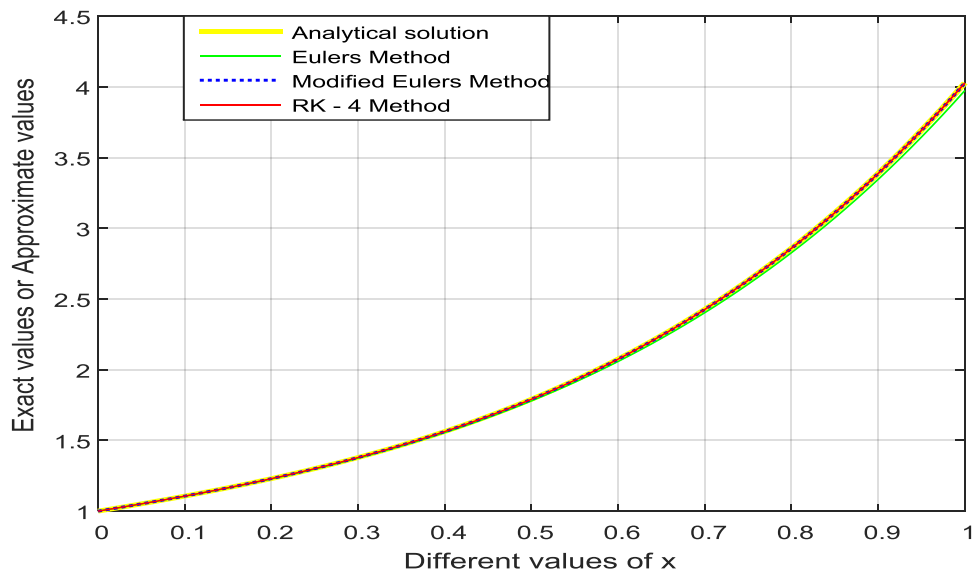
**Fig - 3.1.6 Analytical and Approximate solutions of step length  $h = 0.1$**



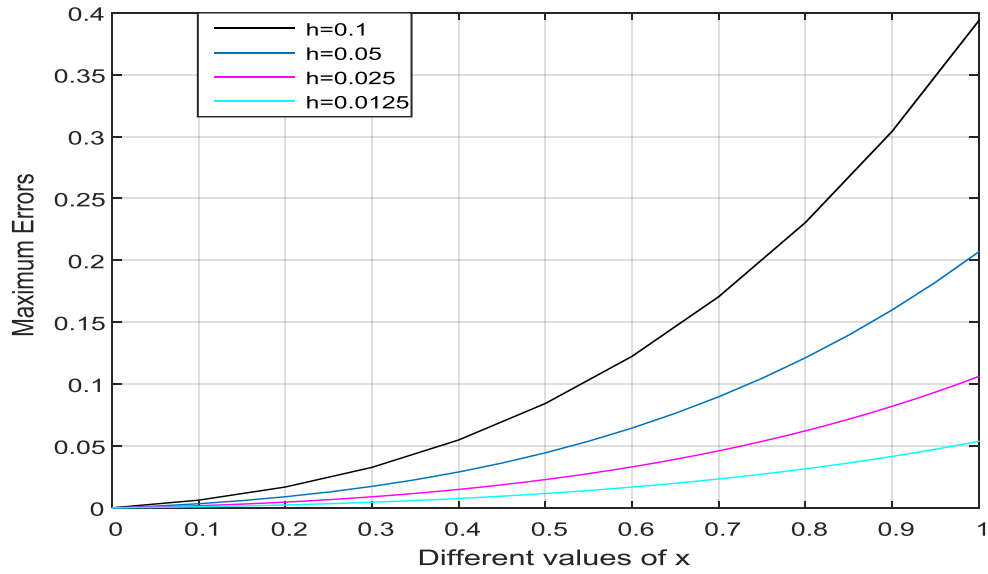
**Fig - 3.1.7 Analytical and Approximate solutions of step length  $h = 0.05$**



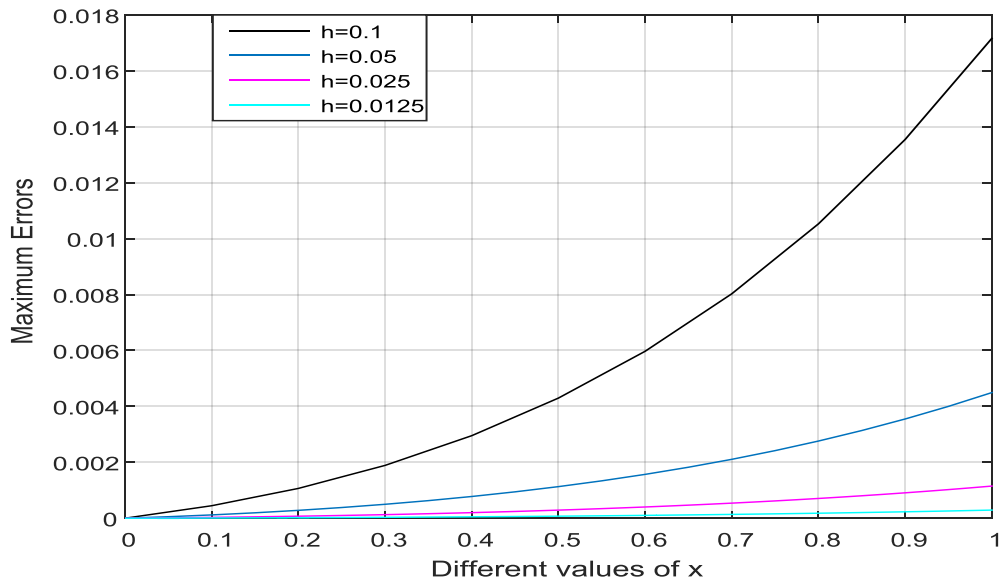
**Fig - 3.1.8 Analytical and Approximate solutions of step length  $h = 0.025$**



**Fig - 3.1.9 Analytical and Approximate solutions of step length  $h = 0.0125$**



**Fig - 3.1.10 Errors of various step length by EM**



**Fig - 3.1.11 Errors of various step length by Modified EM**

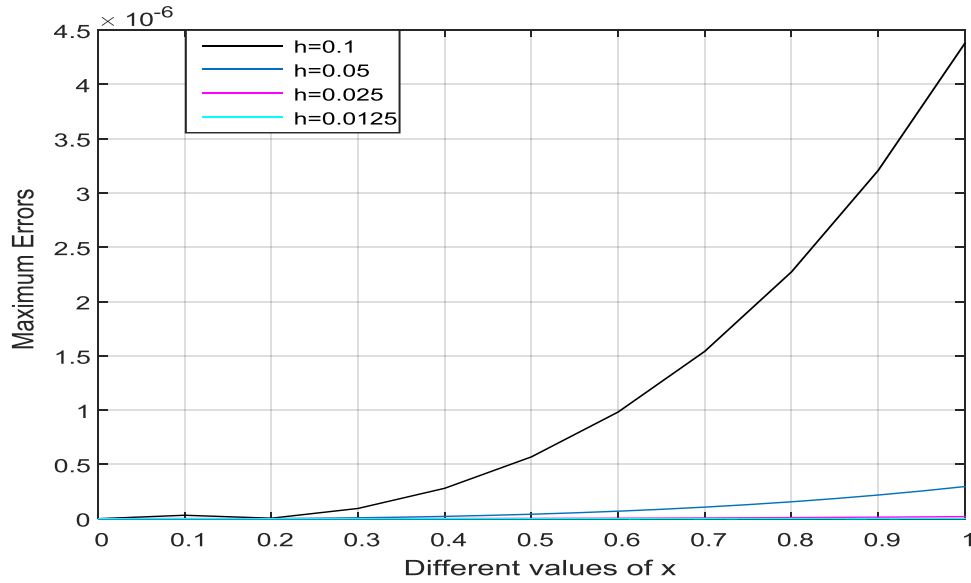


Fig - 3.1.12 Errors of various step length by RK-4 Method

**Numerical Example (3.2)**

We look at the (IVP)  $y'(x) = 2e^x - y$ ;  $y(0) = 2$  in the range  $0 \leq x \leq 1$ . Then the analytical solution of the problem is known as  $y(x) = e^x + e^{-x}$ . The numerical solutions and utmost errors are acquired and demonstrated as tabular form in 5.2. (1) – (4) and approximate solutions are represented by graphs in fig – 5.2. (1) – (12).

**Table (Example 3.2)** (1) Approximate solutions and utmost errors of step length  $h = 0.1$  (2) Approximate solutions and utmost errors of step length  $h = 0.05$  (3) Approximate solutions and utmost errors of step length  $h = 0.025$  (4) Approximate solutions and utmost errors of step length  $h = 0.0125$ .

**3.2. (1) Approximate solutions and utmost errors of step length  $h = 0.1$**

	Eulers Method	Errors	Modified Eulers Method	Errors	RK-4 Method	Errors	Exact Solution
$x_n$	$y(x_n)$	$E_R$	$y(x_n)$	$E_R$	$y(x_n)$	$E_R$	$y_n$
0	2.000000000000000	0	2.000000000000000	0	2.000000000000000	0	2.000000000000000
0.1	2.000000000000000	1.00083e-02	2.010254219275205	2.45883e-04	2.010008581326214	2.45215e-07	2.010008336111607
0.2	2.021034183615130	1.90993e-02	2.040595207808960	4.61697e-04	2.040133987701432	4.76463e-07	2.040133511238152
0.3	2.063211316885651	2.74657e-02	2.091329718823056	6.52691e-04	2.090677725880737	6.97623e-07	2.090677028257721
0.4	2.126861946712286	3.52828e-02	2.162968317177757	8.23574e-04	2.162145655998045	9.12321e-07	2.162144743676910
0.5	2.212540691569312	4.27112e-02	2.256230517167491	9.78587e-04	2.255253054398503	1.12399e-06	2.255251930412761
0.6	2.321030876552406	4.98996e-02	2.372052008903057	1.12157e-03	2.370931772377368	1.33589e-06	2.370930436484535
0.7	2.453351548975268	5.69865e-02	2.511594045856141	1.25603e-03	2.510339562472562	1.55121e-06	2.510338011261886
0.8	2.610766935571836	6.41030e-02	2.676255087747637	1.38520e-03	2.674871665651688	1.77304e-06	2.674869892609689
0.9	2.794798427713146	7.13743e-02	2.867684815511885	1.51204e-03	2.866174775362072	2.00446e-06	2.866172770897549
1.0	3.007239207173221	7.89221e-02	3.087800658789856	1.63939e-03	3.086163518200849	2.24857e-06	3.086161269630487

**3.2. (2) Approximate solutions and utmost errors of step length  $h = 0.05$**

	Eulers Method	Errors	Modified Eulers Method	Errors	RK-4 Method	Errors	Exact Solution
$x_n$	$y(x_n)$	$E_R$	$y(x_n)$	$E_R$	$y(x_n)$	$E_R$	$y_n$
0	2.000000000000000	0	2.000000000000000	0	2.000000000000000	0	2.000000000000000
0.1	2.005127109637602	4.88123e-03	2.010068338187410	6.00021e-05	2.010008351129569	1.50180e-08	2.010008336111607
0.2	2.030801877937951	9.33163e-03	2.040246276311959	1.12765e-04	2.040133540440076	2.92019e-08	2.040133511238152
0.3	2.077234498532991	1.34425e-02	2.090836591051135	1.59563e-04	2.090677071045236	4.27875e-08	2.090677028257721
0.4	2.144847476505071	1.72973e-02	2.162346282072059	2.01538e-04	2.162144799672367	5.59955e-08	2.162144743676910
0.5	2.234279412370764	2.09725e-02	2.255491653002273	2.39723e-04	2.255251999447391	6.90346e-08	2.255251930412761

0.6	2.346390992167867	2.45394e-02	2.371205486543052	2.75050e-04	2.370930518589462	8.21049e-08	2.370930436484535
0.7	2.482273239369988	2.80648e-02	2.510646385597229	3.08374e-04	2.510338106661832	9.53999e-08	2.510338011261886
0.8	2.643258107402377	3.16118e-02	2.675210373955941	3.40481e-04	2.674870001719252	1.09110e-07	2.674869892609689
0.9	2.830931515330029	3.52413e-02	2.866544872697865	3.72102e-04	2.866172894319943	1.23422e-07	2.866172770897549
1.0	3.047148954076846	3.90123e-02	3.086565192226665	4.03923e-04	3.086161408158689	1.38528e-07	3.086161269630487

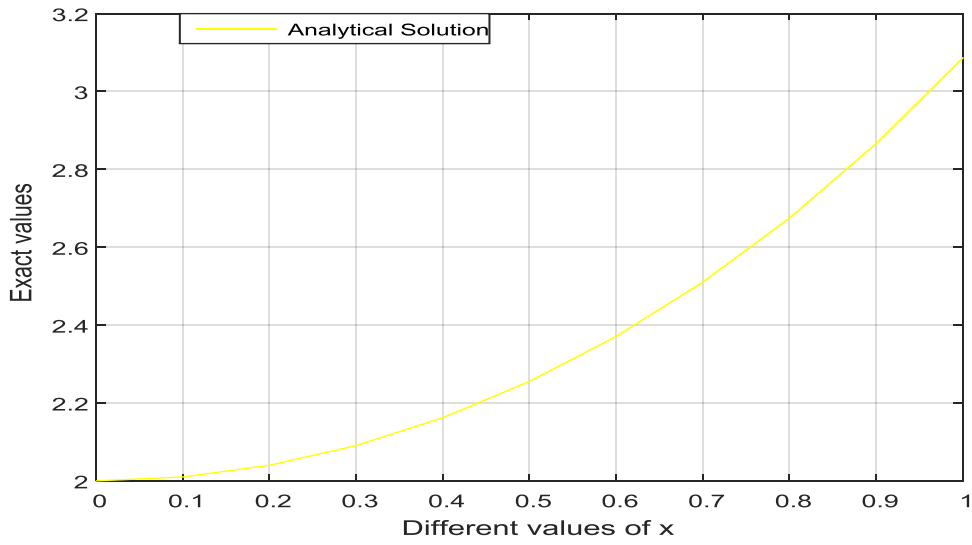
### 3.2. (3) Approximate solutions and utmost errors of step length $h = 0.025$

	Eulers Method	Errors	Modified Eulers Method	Errors	RK-4 Method	Errors	Exact Solution
$x_n$	$y(x_n)$	$E_R$	$y(x_n)$	$E_R$	$y(x_n)$	$E_R$	$y_n$
0	2.000000000000000	0	2.000000000000000	0	2.000000000000000	0	2.000000000000000
0.1	2.007596932814989	2.41140e-03	2.010023156416876	1.48203e-05	2.010008337040653	9.29046e-10	2.010008336111607
0.2	2.035519631334040	4.61388e-03	2.040161376327976	2.78651e-05	2.040133513045292	1.80714e-09	2.040133511238152
0.3	2.084025108729520	6.65192e-03	2.090716476227776	3.94480e-05	2.090677030906527	2.64881e-09	2.090677028257721
0.4	2.153578539420328	8.56620e-03	2.162194594189345	4.98505e-05	2.162144747144563	3.46765e-09	2.162144743676910
0.5	2.244857706478852	1.03942e-02	2.255311257121738	5.93267e-05	2.255251934689333	4.27657e-09	2.255251930412761
0.6	2.358759596950422	1.21708e-02	2.370998543503528	6.81070e-05	2.370930441572441	5.08791e-09	2.370930436484535
0.7	2.496409209162007	1.39288e-02	2.510414413294527	7.64020e-05	2.510338017175503	5.91362e-09	2.510338011261886
0.8	2.659170658421293	1.56992e-02	2.674954298416170	8.44058e-05	2.674869899375139	6.76545e-09	2.674869892609689
0.9	2.848660690678408	1.75121e-02	2.866265069814266	9.22989e-05	2.866172778552622	7.65507e-09	2.866172770897549
1.0	3.066764737974203	1.93965e-02	3.086261520902300	1.00251e-04	3.086161278224727	8.59424e-09	3.086161269630487

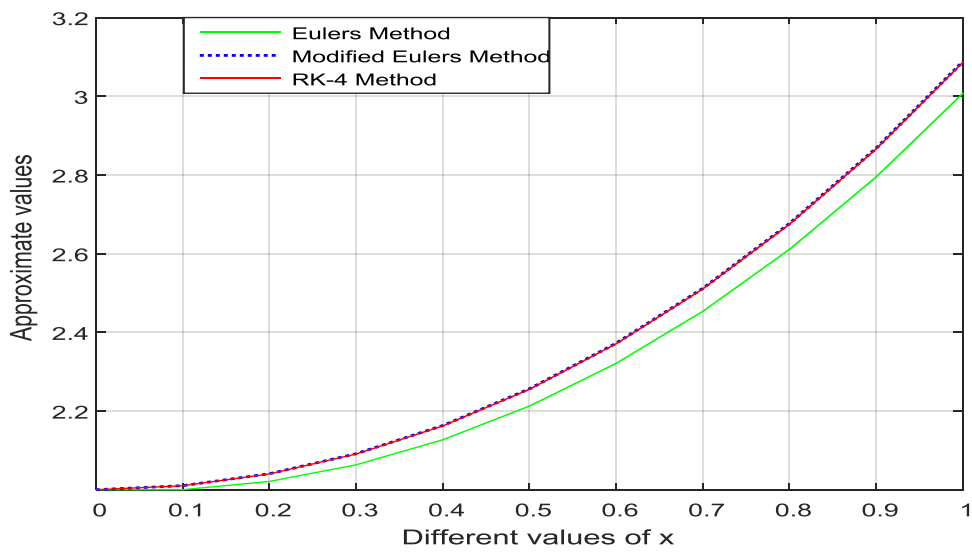
### 3.2. (4) Approximate solutions and utmost errors of step length $h = 0.0125$

	Eulers Method	Errors	Modified Eulers Method	Errors	RK-4 Method	Errors	Exact Solution
$x_n$	$y(x_n)$	$E_R$	$y(x_n)$	$E_R$	$y(x_n)$	$E_R$	$y_n$
0	2.000000000000000	0	2.000000000000000	0	2.000000000000000	0	2.000000000000000
0.1	2.008809759391109	1.19858e-03	2.010012018877750	3.68277e-06	2.010008336169374	5.77662e-11	2.010008336111607
0.2	2.037839255684026	2.29426e-03	2.040140437133956	6.92590e-06	2.040133511350536	1.12384e-10	2.040133511238152
0.3	2.087368026757855	3.30900e-03	2.090686835448384	9.80719e-06	2.090677028422476	1.64755e-10	2.090677028257721
0.4	2.157881827094896	4.26292e-03	2.162157140193059	1.23965e-05	2.162144743892634	2.15724e-10	2.162144743676910
0.5	2.250077388571024	5.17454e-03	2.255266687324053	1.47569e-05	2.255251930678854	2.66092e-10	2.255251930412761
0.6	2.364869302428935	6.06113e-03	2.370947382075694	1.69456e-05	2.370930436801160	3.16625e-10	2.370930436484535
0.7	2.503399090387549	6.93892e-03	2.510357026117554	1.90149e-05	2.510338011629952	3.68066e-10	2.510338011261886
0.8	2.667046554838857	7.82334e-03	2.674890905526894	2.10129e-05	2.674869893030834	4.21147e-10	2.674869892609689
0.9	2.857443520973383	8.72925e-03	2.866195755555938	2.29847e-05	2.866172771374138	4.76592e-10	2.866172770897549
1.0	3.076490107686969	9.67116e-03	3.086186241960812	2.49723e-05	3.086161270165616	5.35132e-10	3.086161269630487

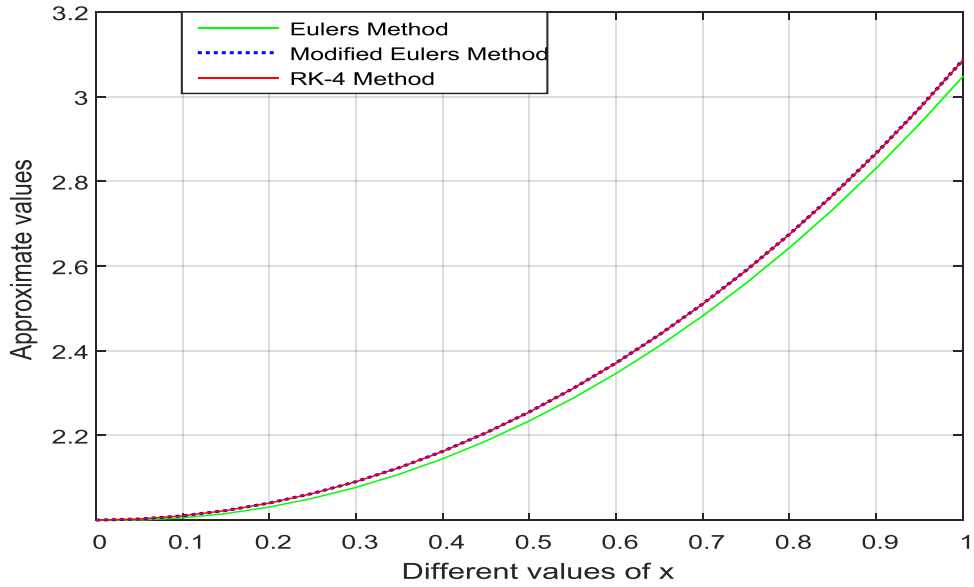
**MATLAB Graphs of Example (3.2)**



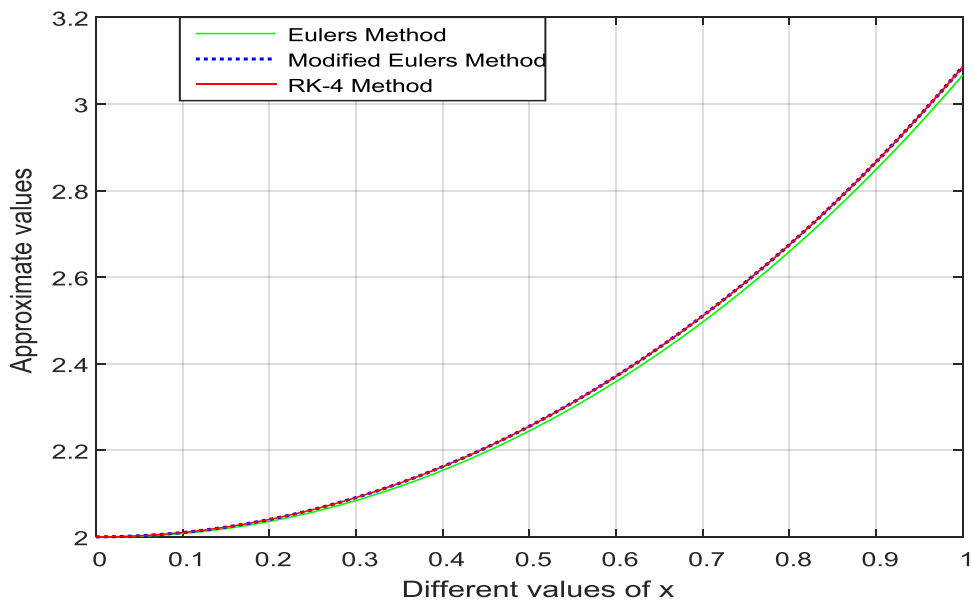
**Fig - 3.2.1 Analytical solution**



**Fig – 3.2.2 Approximate solutions of step length  $h = 0.1$**

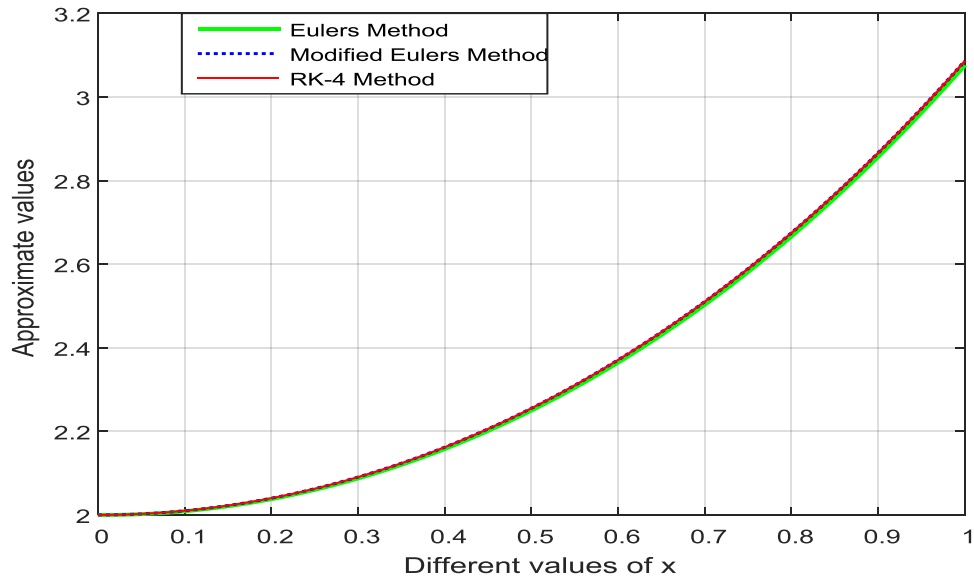


**Fig – 3.2.3** Approximate solutions of step length  $h = 0.05$

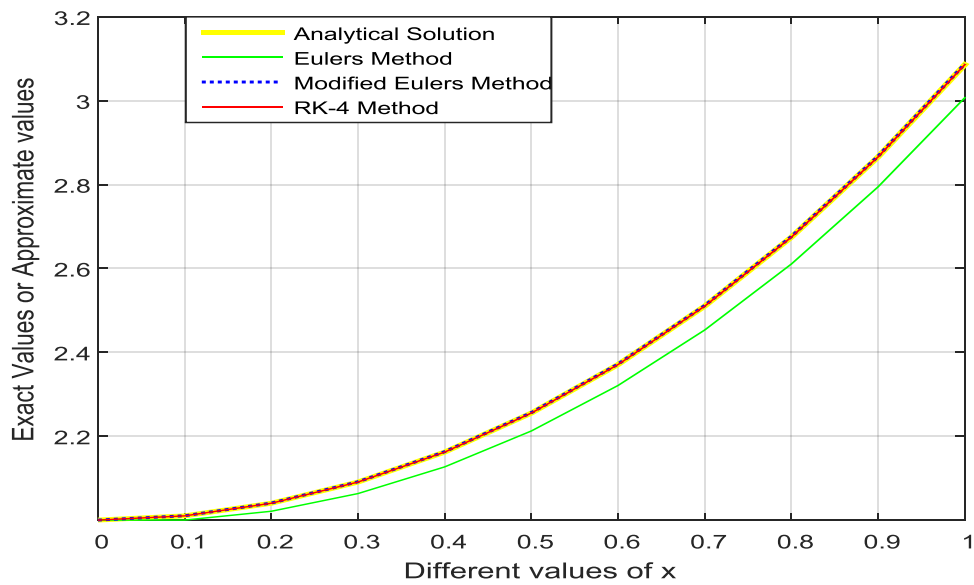


**Fig – 3.2.4** Approximate solutions of step length  $h = 0.025$

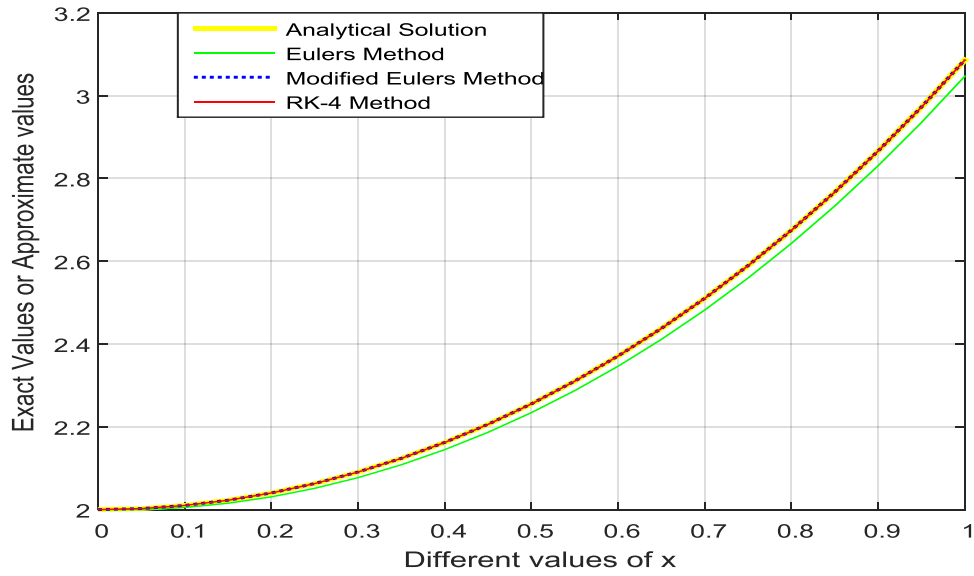




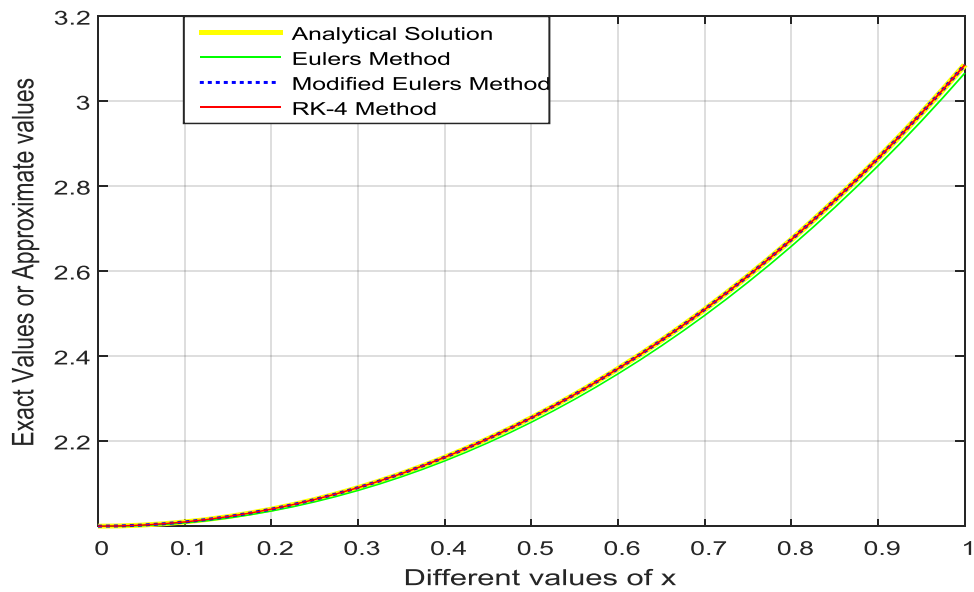
**Fig - 3.2.5** Approximate solutions of step length  $h = 0.0125$



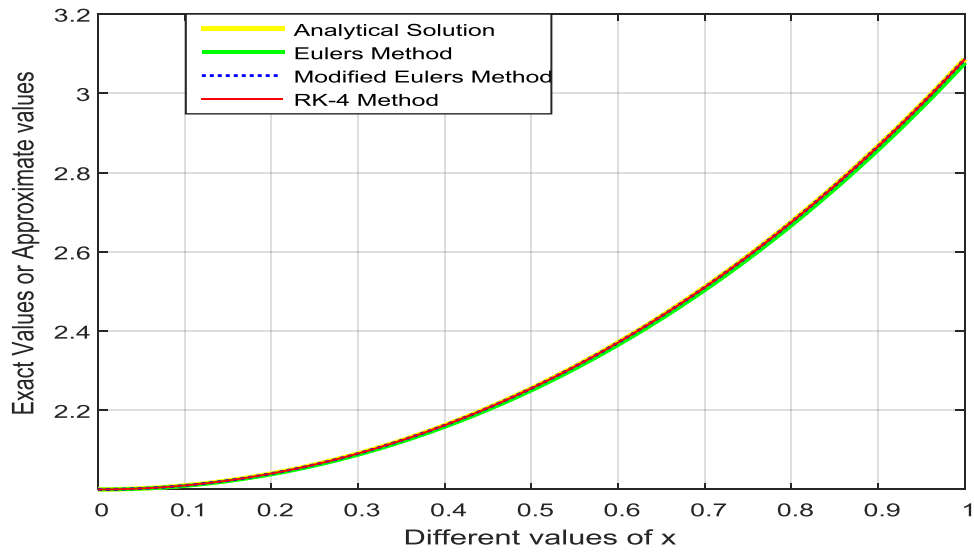
**Fig - 3.2.6** Analytical and Approximate solutions of step length  $h = 0.1$



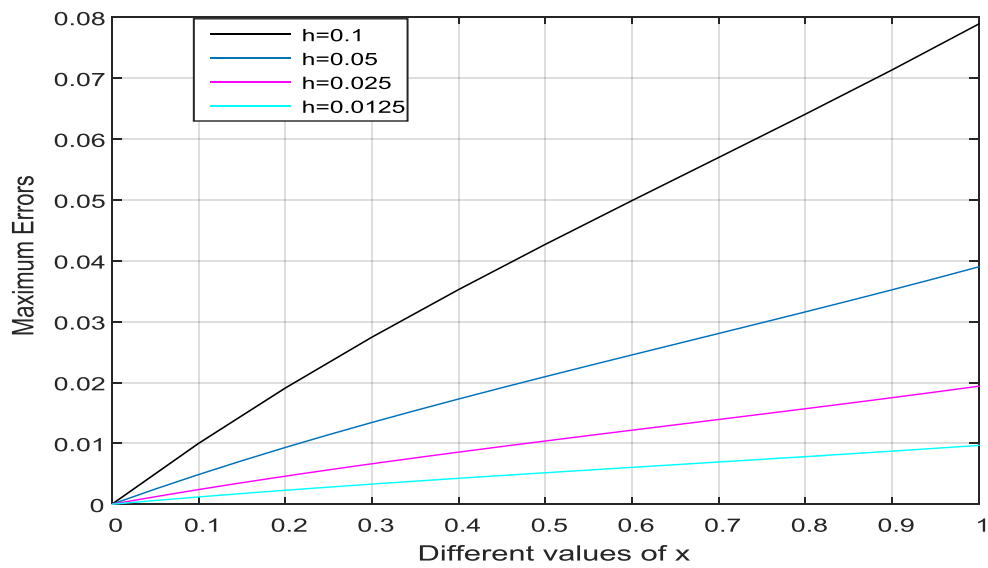
**Fig - 3.2.7 Analytical and Approximate solutions of step length  $h = 0.05$**



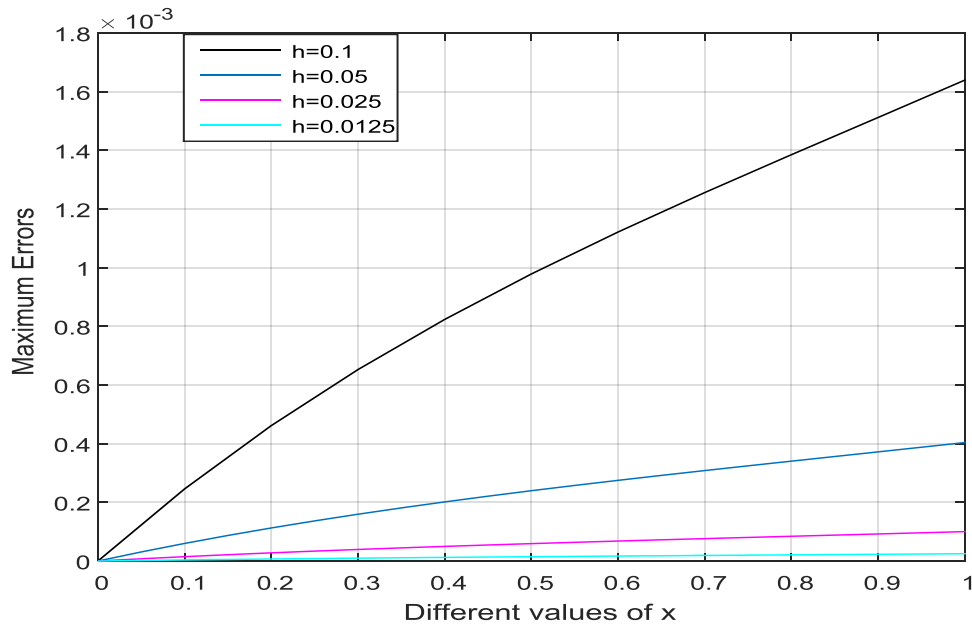
**Fig - 3.2.8 Analytical and Approximate solutions of step length  $h = 0.025$**



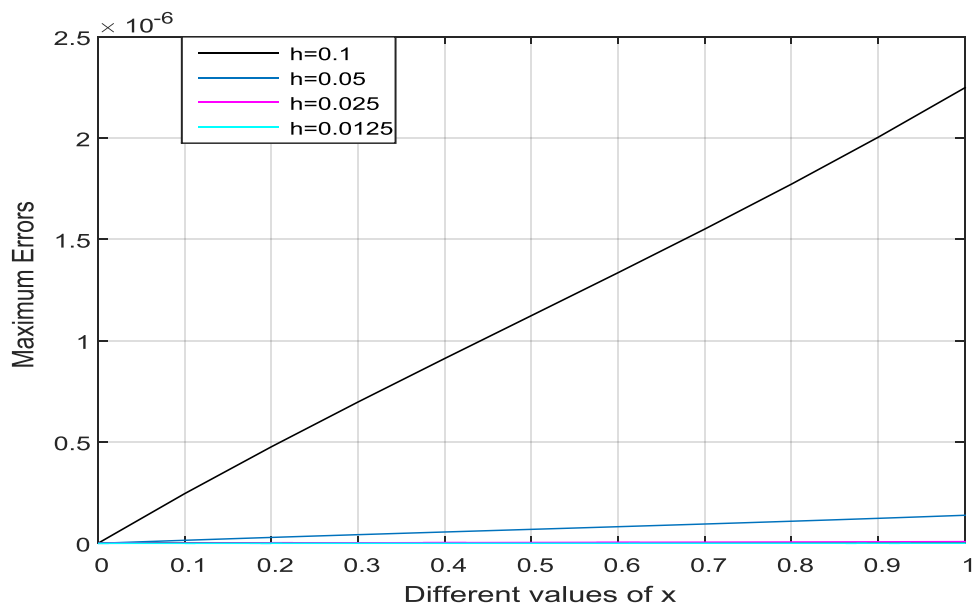
**Fig - 3.2.9 Analytical and Approximate solutions of step length  $h = 0.0125$**



**Fig - 3.2.10 Errors of various step length by EM**



**Fig - 3.2.11 Errors of various step length by Modified EM**



**Fig - 3.2.12 Errors of various step length by RK-4 Method**

**Numerical Example (3.3)**

We look at the (IVP)  $y'(x) = x + y; y(0) = 1$  in the range  $0 \leq x \leq 1$ . Then the analytical solution of the problem is known as  $y(x) = -(x + 1) + 2e^x$ . The numerical solutions and utmost errors are acquired and demonstrated as tabular form in 5.3. (1) – (4) and approximate solutions are represented by graphs in fig – 5.3. (1) – (12).

Table (Example 3.3) (1) Approximate solutions and utmost errors of step length  $h = 0.1$  (2) Approximate solutions and utmost errors of step length  $h = 0.05$  (3) Approximate solutions and utmost errors of step length  $h = 0.025$  (4) Approximate solutions and utmost errors of step length  $h = 0.0125$ .

**3.3. (1) Approximate solutions and utmost errors of step length  $h = 0.1$** 

	Eulers Method	Errors	Modified Eulers Method	Errors	RK-4 Method	Errors	Exact Solution
$x_n$	$y(x_n)$	$E_R$	$y(x_n)$	$E_R$	$y(x_n)$	$E_R$	$y_n$
0	1.0000000000000000	0	1.0000000000000000	0	1.0000000000000000	0	1.0000000000000000
0.1	1.1000000000000000	1.03418e-02	1.1100000000000000	3.41836e-04	1.1103416666666667	1.69485e-07	1.110341836151295
0.2	1.2200000000000000	2.28055e-02	1.2420500000000000	7.55516e-04	1.242805141701389	3.74619e-07	1.242805516320340
0.3	1.3620000000000000	3.77176e-02	1.3984652500000000	1.25237e-03	1.399716994125076	6.21027e-07	1.399717615152006
0.4	1.5282000000000000	5.54494e-02	1.5818041012500000	1.84529e-03	1.583648480161372	9.15121e-07	1.583649395282541
0.5	1.7210200000000000	7.64225e-02	1.794893531881250	2.54901e-03	1.797441277193676	1.26421e-06	1.797442541400256
0.6	1.9431220000000000	1.01116e-01	2.040857352728782	3.38025e-03	2.044235924183866	1.67660e-06	2.044237600781018
0.7	2.1974342000000000	1.30071e-01	2.323147374765304	4.35804e-03	2.327503253193554	2.16175e-06	2.327505414940953
0.8	2.4871776200000000	1.63904e-01	2.645577849115661	5.50401e-03	2.651079126584631	2.73040e-06	2.651081856984935
0.9	2.8158953820000000	2.03311e-01	3.012363523272805	6.84270e-03	3.019202827560142	3.39475e-06	3.019206222313899
1.0	3.1874849202000000	2.49079e-01	3.428161693216450	8.40196e-03	3.436559488270332	4.16865e-06	3.436563656918090

**3.3. (2) Approximate solutions and utmost errors of step length  $h = 0.05$** 

	Eulers Method	Errors	Modified Eulers Method	Errors	RK-4 Method	Errors	Exact Solution
$x_n$	$y(x_n)$	$E_R$	$y(x_n)$	$E_R$	$y(x_n)$	$E_R$	$y_n$
0	1.0000000000000000	0	1.0000000000000000	0	1.0000000000000000	0	1.0000000000000000
0.1	1.1050000000000000	5.34184e-03	1.1102531250000000	8.87112e-05	1.110341825108643	1.10427e-08	1.110341836151295
0.2	1.2310125000000000	1.17930e-02	1.242609438286133	1.96078e-04	1.242805491912302	2.44080e-08	1.242805516320340
0.3	1.3801912812500000	1.95263e-02	1.399392572063210	3.25043e-04	1.399717574689427	4.04626e-08	1.399717615152006
0.4	1.554910887578125	2.87385e-02	1.583170434002249	4.78961e-04	1.583649335658453	5.96241e-08	1.583649395282541
0.5	1.757789253554883	3.96533e-02	1.796780887080538	6.61654e-04	1.797442459031747	8.23685e-08	1.797442541400256
0.6	1.991712652044258	5.25249e-02	2.043360129055016	8.77472e-04	2.044237491543481	1.09238e-07	2.044237600781018
0.7	2.259863198878795	6.76422e-02	2.326374055372126	1.13136e-03	2.327505274093780	1.40847e-07	2.327505414940953
0.8	2.565749176763871	8.53327e-02	2.649652919152582	1.42894e-03	2.651081679087565	1.77897e-07	2.651081856984935
0.9	2.913238467382168	1.05968e-01	3.017429634861184	1.77659e-03	3.019206001131025	2.21183e-07	3.019206222313899
1.0	3.306595410288840	1.29968e-01	3.434382108709771	2.18155e-03	3.436563385312669	2.71605e-07	3.436563656918090

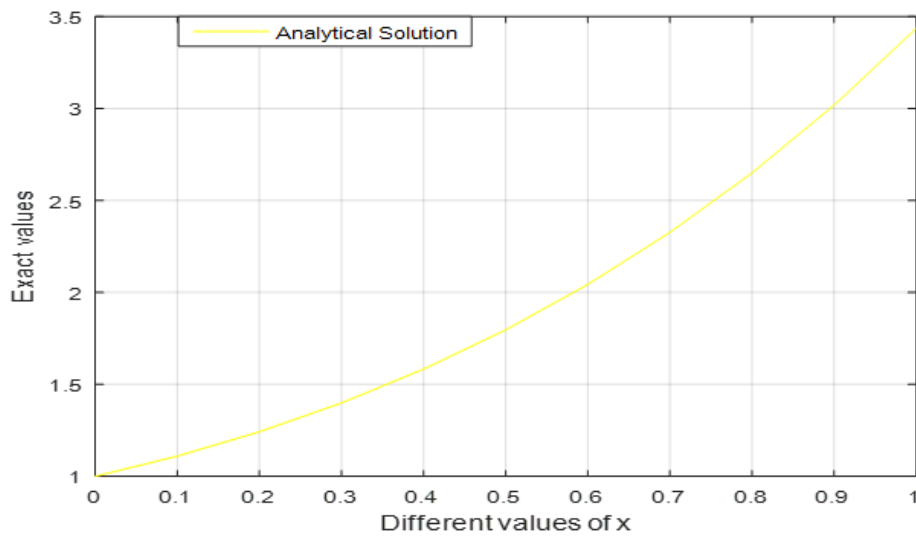
**3.3. (3) Approximate solutions and utmost errors of step length  $h = 0.025$** 

	Eulers Method	Errors	Modified Eulers Method	Errors	RK-4 Method	Errors	Exact Solution
$x_n$	$y(x_n)$	$E_R$	$y(x_n)$	$E_R$	$y(x_n)$	$E_R$	$y_n$
0	1.0000000000000000	0	1.0000000000000000	0	1.0000000000000000	0	1.0000000000000000
0.1	1.1076257812500000	2.71605e-03	1.110319239263935	2.25969e-05	1.110341835446614	7.04682e-10	1.110341836151295
0.2	1.236805795019837	5.99972e-03	1.242755569730151	4.99466e-05	1.242805514762752	1.55759e-09	1.242805516320340
0.3	1.389777648492597	9.93997e-03	1.399634816296844	8.27989e-05	1.399717612569905	2.58210e-09	1.399717615152006
0.4	1.569011241321129	1.46382e-02	1.583527386723836	1.22009e-04	1.583649391477657	3.80488e-09	1.583649395282541
0.5	1.777232880580795	2.02097e-02	1.797273991873273	1.68550e-04	1.797442536143947	5.25631e-09	1.797442541400256
0.6	2.017451899165182	2.67857e-02	2.044014070681046	2.23530e-04	2.044237593810074	6.97094e-09	2.044237600781018
0.7	2.292990037514416	3.45154e-02	2.327217204287402	2.88211e-04	2.327505405952854	8.98810e-09	2.327505414940953
0.8	2.607513875545614	4.35680e-02	2.650717833665581	3.64023e-04	2.651081845632495	1.13524e-08	2.651081856984935
0.9	2.965070631435801	5.41356e-02	3.018753628143069	4.52594e-04	3.019206208199214	1.41147e-08	3.019206222313899
1.0	3.3701276779946	6.64360e-02	3.436007888741954	5.55768e-04	3.436563639585713	1.73324e-08	3.436563656918090

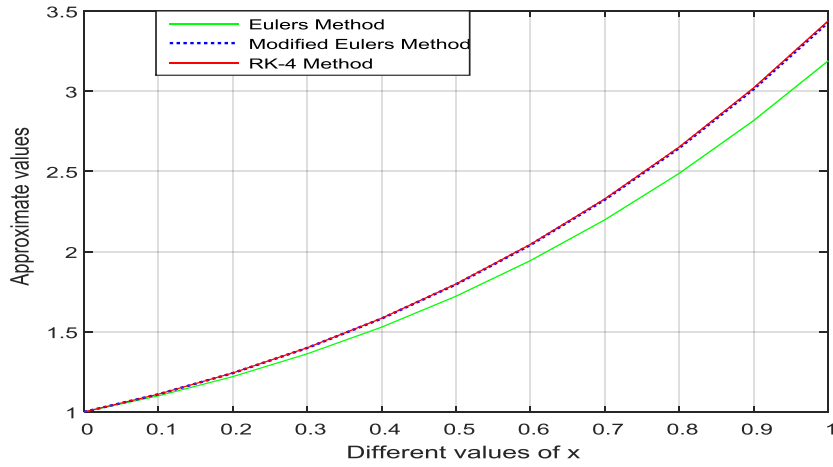
**3.3. (4) Approximate solutions and utmost errors of step length  $h = 0.0125$**

	Eulers Method	Errors	Modified Eulers Method	Errors	RK-4 Method	Errors	Exact Solution
$x_n$	$y(x_n)$	$E_R$	$y(x_n)$	$E_R$	$y(x_n)$	$E_R$	$y_n$
0	1.000000000000000	0	1.000000000000000	0	1.000000000000000	0	1.000000000000000
0.1	1.108972202362824	1.36963e-03	1.110336133753741	5.70240e-06	1.110341836106792	4.45035e-11	1.110341836151295
0.2	1.239779095405834	3.02642e-03	1.242792912088718	1.26042e-05	1.242805516221972	9.83680e-11	1.242805516320340
0.3	1.394702100828703	5.01551e-03	1.399696720433609	2.08947e-05	1.399717614988936	1.63070e-10	1.399717615152006
0.4	1.576261017189655	7.38838e-03	1.583618605675438	3.07896e-05	1.583649395042247	2.40294e-10	1.583649395282541
0.5	1.787238926974026	1.02036e-02	1.797400006732188	4.25347e-05	1.797442541068299	3.31958e-10	1.797442541400256
0.6	2.030709706105312	1.35279e-02	2.044181191159992	5.64096e-05	2.044237600340775	4.40243e-10	2.044237600781018
0.7	2.310068407817767	1.74370e-02	2.327432682383340	7.27326e-05	2.327505414373319	5.67633e-10	2.327505414940953
0.8	2.629064821221400	2.20170e-02	2.650989992066324	9.18649e-05	2.651081856267983	7.16950e-10	2.651081856984935
0.9	2.991840536270573	2.73657e-02	3.019092005220236	1.14217e-04	3.019206221422500	8.91395e-10	3.019206222313899
1.0	3.402969881506675	3.35938e-02	3.436423402198716	1.40255e-04	3.436563655823480	1.09460e-09	3.436563656918090

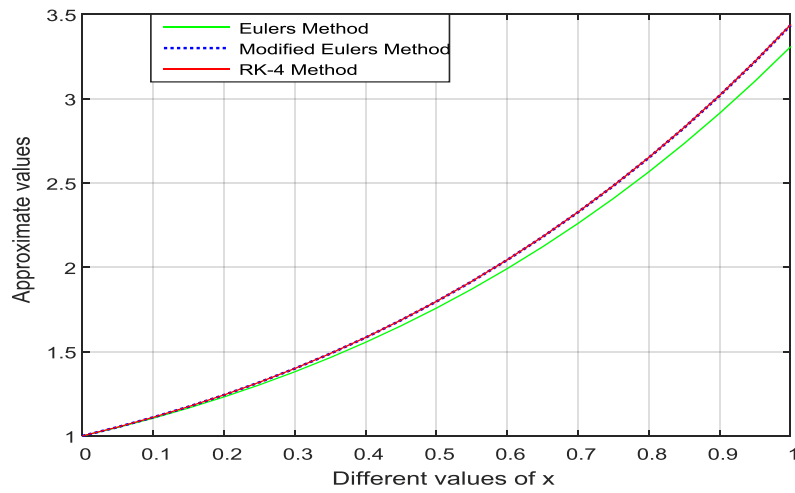
**MATLAB Graphs of Example (3.3)**



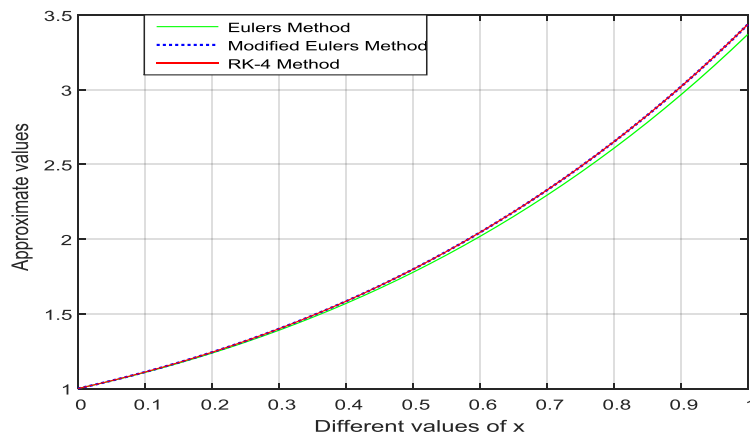
**Fig – 3.3.1 Analytical solution**



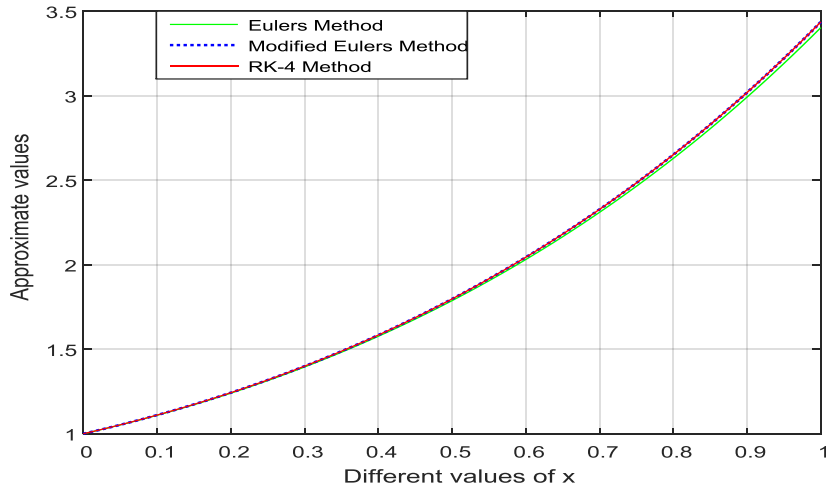
**Fig – 3.3.2** Approximate solutions of step length  $h = 0.1$



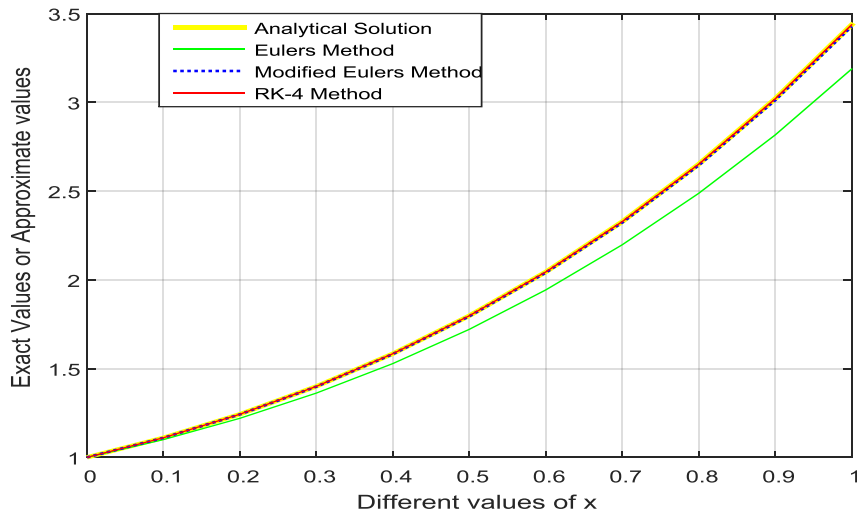
**Fig - 3.3.3** Approximate solutions of step length  $h = 0.05$



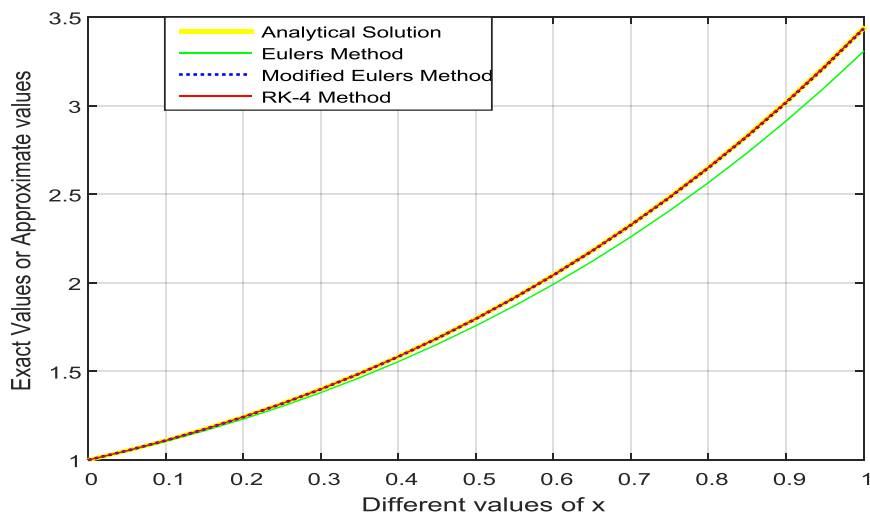
**Fig – 3.3.4** Approximate solutions of step length  $h = 0.025$



**Fig – 3.3.5** Approximate solutions of step length  $h = 0.0125$

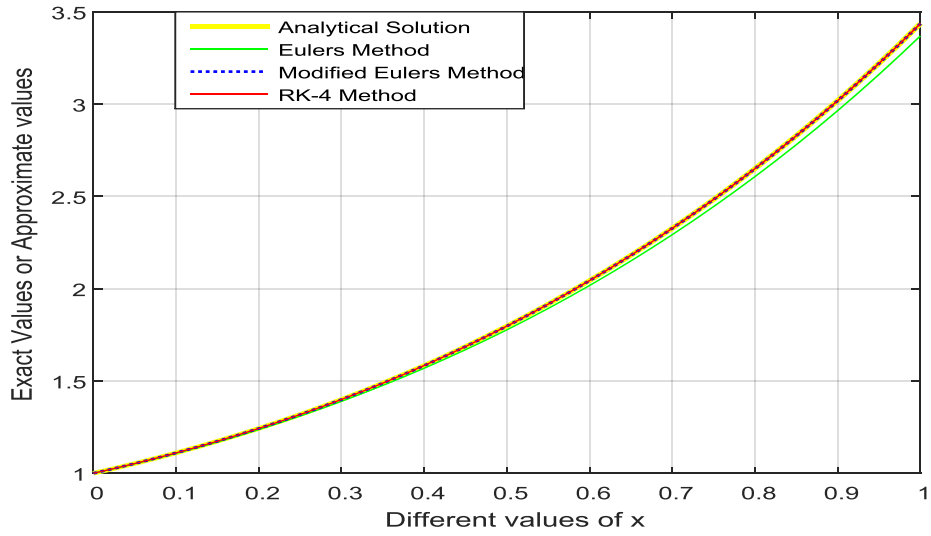


**Fig - 3.3.6** Analytical and Approximate solutions of step length  $h = 0.1$

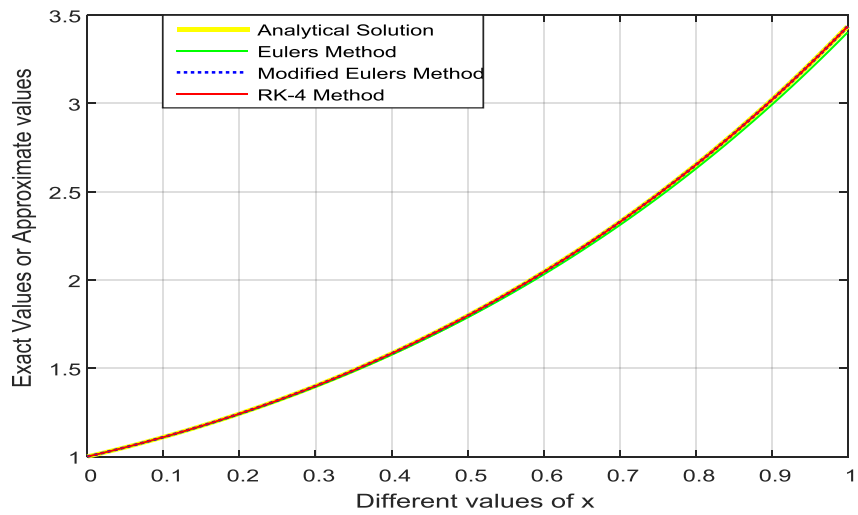


**Fig - 3.3.7** Analytical and Approximate solutions of step length  $h = 0.05$

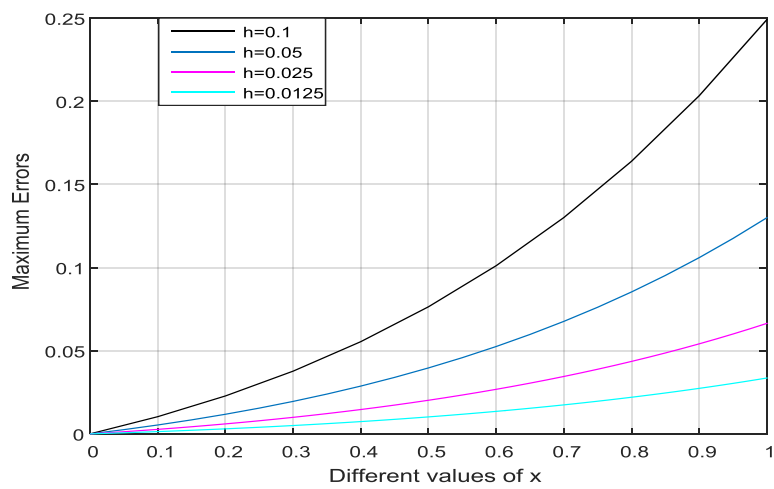




**Fig - 3.3.8 Analytical and Approximate solutions of step length  $h = 0.025$**



**Fig - 3.3.9 Analytical and Approximate solutions of step length  $h = 0.0125$**



**Fig - 3.3.10 Errors of various step length by EM**

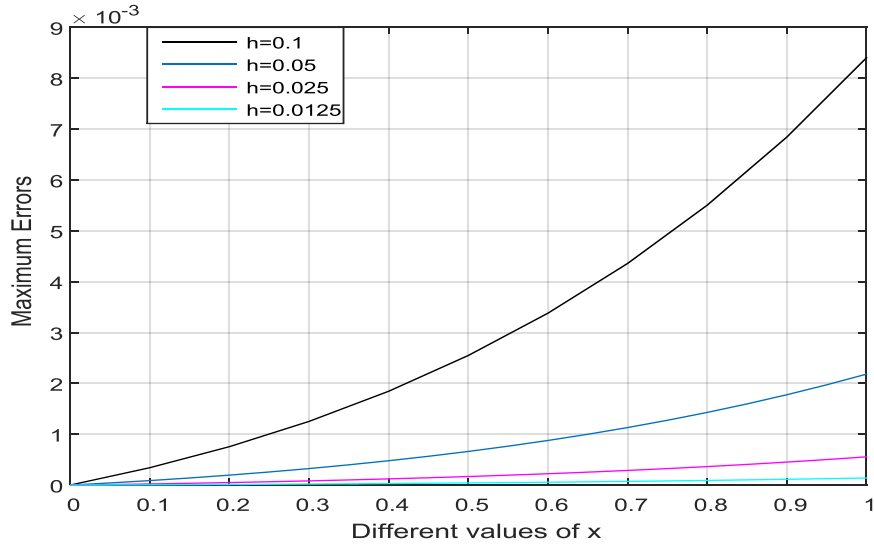


Fig - 3.3.11 Errors of various step length by Modified EM

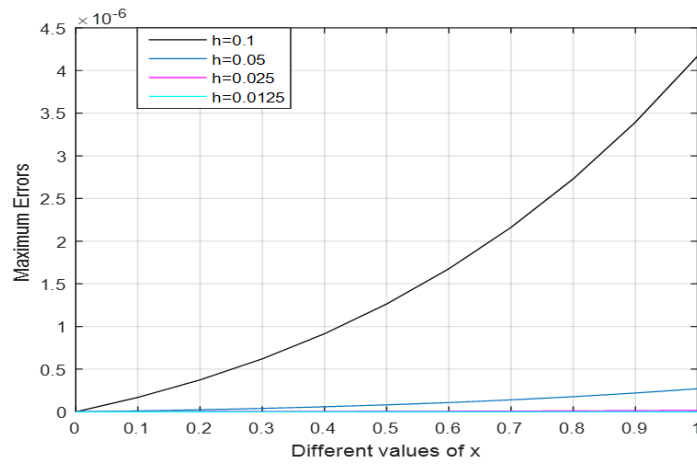


Fig - 3.3.12 Errors of various step length by RK-4 Method

#### IV. DISCUSSION OF RESULTS

The outcomes which have been obtained demonstrated in Tables 5.1 (1) – (4), Tables 5.2 (1) – (4) and Tables 5.3 (1) – (4) and graphs are exhibited in fig - 5.1.1 – 12, 5.2.1 – 12 and 5.3.1 – 12. The numerical approximate solution is computed with step sizes  $h = 0.1, 0.05, 0.025, 0.0125$  as well as utmost errors are evaluated at particular step length on the interval  $[0, 1]$ . For every method from above tables, we state that an approximate solution is convergent to analytical solution if step length leads to decrease errors in such a way that in the limiting case when the step length goes to 0, the error also tends to 0.

We observe that the Euler’s Method, Modified Euler’s Method iterations employing step length 0.1 and 0.05 is not converging to analytical solution although for step length 0.025 and 0.0125 converges gradually to exactitude solution. While, we observe that the RK-4 Method converges from starting to exactitude solution. Therefore, it is shown that tiny step length gives the improved estimation. We see that  $EM < MEM < RK-4 \text{ Method} \approx \text{Analytical Solution}$ . Also the error values for each  $x$  are in the order,  $RK-4 \text{ Method} < MEM < EM$ .

Figures - 5.1.1 – 12, 5.2.1 – 12 and 5.3.1 – 12 show the graphs of the Numerical approximate, Analytical Solution and maximum errors on the interval  $[0, 1]$  using MATLAB software. Where Yellow, Green, Dotted Blue and Red color curves denote Analytical Solution, EM, MEM and RK-4 Methods curves respectively. In each fig – 5.1.5, 5.2.5 and 5.3.5, the Green curve i.e., Euler’s curve is below to the other curves and the Dotted Blue curve i.e., Modified Euler’s curve is in between Euler’s curve and RK-4 curve.

Also, in each fig – 5.1.6 – 7, 5.2.6 – 7 and 5.3.6 – 7 as we go on decreasing the step size for  $h = 0.1, 0.05, 0.025, 0.0125$ . We see that firstly Red curve RK-4 Curve becomes nearest to Analytical Solution, secondly Dotted Blue curve Modified Euler’s curve gets nearer to Analytical Solution and thirdly Green curve Euler’s curve becomes near to Analytical Solution. Therefore, the RK-4 Curve is more nearest to Analytical Solution.

Also, in each fig – 5.1.10 – 12, 5.2.10 – 12 and 5.3.10 – 12, where Black, Sky, Magenta and Cyan color curves denote error curves for step sizes  $h = 0.1, 0.05, 0.025, 0.0125$  respectively. We see that error for step size  $h = 0.1, 0.05, 0.025, 0.0125$  using RK-4 Method is becoming extreme less. Therefore, RK-4 Method gives smaller errors as compared to that of the EM and Modified EM.

## V. CONCLUSION

In this work, EM, MEM and RK-4 Method are employed to solve ODEs along with initial value problems (IVP). In order to find better consequences, the step length needs to be very small for three proposed methods. We can look at the figures above, the correctness of the numerical methods for minimizing the step length  $h$ . The approximate values acquired by using three suggested methods are in accordance with that of the corrected values. After making comparison of acquired outcomes of three standard methods under examination, the convergent rates of EM, MEM are  $O(h)$  and  $O(h^2)$  respectively and convergent rate of RK-4 Method is  $O(h^4)$ . The EM, MEM show less correctness due to less accurate numerical outcomes. Above study shows that RK-4 Method is commonly thought to be better than others and numerical solution converges swiftly toward the analytical solution, also the error is very less for RK-4 Method as compared to that of the EM and Modified EM. So we may conclude that the RK-4 Method is emerged to be stronger and more effective and efficient than Euler's Method and Modified Euler's Method in finding approximate solutions of initial value problems (IVP).

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