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Modification of the Variational Iteration Method for the Solution of Linear and Nonlinear Differential Equations

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ABSTRACT

In this work, we have modified a semi-analytical method namely the Variational Iteration method to comfortably solve both linear and nonlinear differential equation of homogeneous and inhomogeneous type. The newly introduced method is simpler and shorter in its computational procedures and time than the other methods. In addition, it does not require linearization, or calculating polynomials for nonlinear part, and the modification is void of the Lagrange's multiplier values. We also compare our results with the exact solutions and other methods such as Differential Transform method, Adomian decomposition method, New Iterative method and the Variational Iteration method. The method is capable of reducing the size of calculations. These advantages make it reliable and its efficiency is demonstrated with numerical examples.

Keywords: Variational Iteration method, Lagrange's multiplier values, Adomian polynomials, and differential equation.

1. INTRODUCTION

The variational iteration method (VIM) was developed by He in (He & Wu, 2007). The method is now used by many researchers in a variety of scientific fields. The method has been proved by many authors [(Araghi et al., 2011), (A. Wazwaz, 2007a), (Okai J. O et al., 2017) and (A. M. Wazwaz, 2009) etc] to be reliable and efficient for a variety of scientific applications, linear and nonlinear as well. It was shown by many authors that this method is powerful over existing techniques such as Adomian method, perturbation method, etc. Unlike the Adomian decomposition method, where computational algorithms are normally used to deal with the nonlinear terms, the VIM is used directly with no requirement or restrictive assumptions for the nonlinear terms (Wazwaz, 2011). The method attacks the nonlinear problem under consideration, without any need to restrictive assumptions that may change the physical structure of the solutions as in the case of linearization [(Wazwaz, 2011) and (Scheiber & Brasov, 2019)]. The Variational Iteration Method is one of the powerful techniques developed to tackle differential problems which could exist in form of partial or ordinary differential equations; the method through its development has seen the formulation of important part such as the Lagrange Multipliers. The Lagrange multiplier play key roles in the application of the method and in most cases are developed to handle specific problems (Goswami & Alqahtani, 2016). The variational iteration method (VIM) has been one of the most often used analytical methods in the past ten years (Wu, 2015). However, the success of the method mainly depends upon accurate identifications of the Lagrange multipliers (Al-saar, 2019) and also It was shown in (Abassy et al., 2007), (Benattia & Belghaba, 2019) and (Al-saar, 2019) that the application of VIM to nonlinear differential equations leads to the calculation of unneeded terms and more time consumed in repeated calculations for series solutions. Insight into the solution procedure of the VIM shows some disadvantages, namely, repeated computations, computations of unneeded terms, failure of the method in the case of inhomogeneous equation and the use of Lagrange Multipliers which consumes time and effort. A modified variational iteration method (MVIM) is introduced to overcome these disadvantages.

2. MATERIALS AND METHOD

2.1 Basics of the Variational Iteration Method (VIM)

The variational iteration method (VIM) established by Ji-Huan He [(J. H. He and Wu, 2007)], is now used to handle a wide variety of linear and nonlinear, homogeneous and inhomogeneous equations. The method provides rapidly convergent successive approximations of the exact solution if such a closed form solution exists, and not components wise as in the case of Adomian's method. The variational iteration method handles linear and nonlinear problems in the same manner without any need to specific restrictions such as the so called Adomian polynomials that we need for nonlinear problems.

To illustrate the basic idea of this method, we consider the following equation:

$$Lu(x) + Nu(x) = g(x) \qquad \dots(1)$$

$$\frac{d^k u(0)}{dt^k} = c_k\,, \quad k = 0,1,2,\dots,n-1. \qquad \dots(2)$$

Where L is a linear operator, N is a nonlinear operator, $g(x)$ is a known continuous function. The basic characteristic of the method is construction of the following correctional functional for Eqn. (1) as proposed by Ji-Huan He in (2006):

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(t)\big(Lu(t) + N\tilde{u}_n(t) - g(t)\big)dt \qquad \text{... (3)}$$

where λ is a general Lagrange's multiplier, which can be identified optimally via the variational theory, and $\tilde{u}_n$ as a restricted variation which means $\delta\tilde{u}_n = 0$. It is to be noted that the Lagrange's multiplier λ can be a constant or a function.

The variational iteration method consists of the following two essential steps.

I. It is required first to determine the Lagrange's multiplier λ that can be identified optimally via integration by parts and using the restricted variation.

II. Having determined λ, an iteration formula, without restricted variation should be used for determination of the successive approximation $u_{n+1}(x), n \geq 0$ of the solution$u(x)$. The zeroth approximation u_0 can be any selective function. Consequently, the result is given by

$$u(x) = \lim_{n\to\infty} u_n(x)\ldots \qquad \text{... (4)}$$

2.2 The Modified Variational Iteration Method (MVIM)

To give a clear overview of our new modification of the variational iteration method, we consider the following differential equation:

$$Lu + Nu = g(x) \qquad \text{...(5)}$$

Where L is a linear operator, N is a nonlinear operator, and $g(x)$ is the source term. According to the MVIM, we can construct the correction functional without considering the Lagrange's multiplier or the integration by part theory. Our modification of the VIM is as follows:

$$u(x) = \sum_{i=0}^{n-1} u^i(0) + \frac{(-1)^q}{(q-1)!}\int_0^x \big(Lu(t) + Nu(t) - g(t)\big)dt \qquad \text{...(6)}$$

Where q is the order of the derivative and $\sum_{i=0}^{n-1} u^i(0)$ is obtained from the given initial condition

$$u_0(x) = \sum_{i=0}^{n-1} u^i(0) \qquad \text{...(7)}$$

$$u_1(x) = \frac{(-1)^q}{(q-1)!}\int_0^x \big(Lu_0(t) + Nu_0(t) - g(t)\big)dt \qquad \text{...(8)}$$

$$\vdots$$

$$u_{n+1}(x) = \frac{(-1)^q}{(q-1)!}\int_0^x \sum_{m=0}^{n} \big(Lu_m(t) + Nu_m(t) - g(t)\big)dt - \sum_{m=0}^{n-1} u_m(x) \qquad \text{...(9)}$$

For $n \geq 1$

The initial approximation u_0 can be any selective function. However, using the initial values $u(0)$, $u'(0)$ and $u''(0)$ are preferably used for the selective initial approximation u_0 as will be seen later. Consequently, the solution is given by

$$u(x) = u_0(x) + \sum_{m=1}^{n-1} u_m(x) \quad \text{...(10)}$$

2.3 The Modified Variational Iteration for Non-homogeneous ODEs

To illustrate the concepts, we consider the following equation

$$u^{\alpha}(x) + au'(x) + bu^{n}(x) = g(x), \quad \text{...(11)}$$

$$u^{\alpha}(0) = \delta_{\alpha},\ \alpha = 1,2,3,... \quad \text{...(12)}$$

Where $n \geq 1$

Where $a(x)$ and $b(x)$ denotes given functions and α represents a parameter describing the order of the derivative $\alpha \geq 2$ and $n \geq 1$.

The MVIM admits the following for Eqn. (11)

$$u(x) = \sum_{k=0}^{n} u^{k}(0)\frac{x^k}{k!} + \frac{(-1)^{\alpha}}{(\alpha-1)!}\int_0^x (x-t)^{\alpha-1}\left(u^{\alpha}(t) + au'(t) + bu^{n}(t) - g(t)\right)dt \quad \text{...(13)}$$

$$u_0(x) = \sum_{k=0}^{n} u^{k}(0)\frac{x^k}{k!} \quad \text{...(14)}$$

$$u_1(x) = \frac{(-1)^{\alpha}}{(\alpha-1)!}\int_0^x (x-t)^{\alpha-1}\left(u_0^{\alpha}(t) + au_0'(t) + bu_0^{\ n}(t) - g(t)\right)dt \quad \text{...(15)}$$

$$u_{n+1}(x) = \frac{(-1)^{\alpha}}{(\alpha-1)!}\int_0^x (x-t)^{\alpha-1}\sum_{m=0}^{n}\left(u_m^{\alpha}(t) + au_m'(t) + bu_m^{n}(t) - g(t)\right)dt \quad \text{...(16)}$$

For $n \geq 1$

Recall that

$$u(x) = \sum_{m=0}^{\infty} u_m(x) \quad \text{...(17)}$$

2.4 MVIM for first order ODEs

We first start our analysis by studying the first order ODE of a standard form

$$u'(x) + p(x)u^{n}(x) = q(x),\ u(0) = c_0 \quad \text{...(18)}$$

Where $n \geq 1$

The MVIM admits the following for Eqn. (18)

$$u(x) = c_0 - \int_0^x \left(u'(t) + p(t)u^{n}(t) - q(t)\right)dt \quad \text{...(19)}$$

$$u_0(x) = c_0 \quad \text{...(20)}$$

$$u_1(x) = -\int_0^x \left(u_0'(t) + p(t)u^n{}_0(t) - q(t)\right)dt \qquad ...(21)$$

$\vdots$

$$u_{n+1}(x) = -\int_0^x \sum_{m=0}^{n} \left(u_m'(t) + p(t)u_m^n(t) - q(t)\right)dt - \sum_{m=0}^{n-1} u_m(x) \qquad ...(22)$$

For $n \geq 1$

Recall that

$$u(x) = u_0(x) + \sum_{m=1}^{n-1} u_m(x) \qquad ...(23)$$

2.5 MVIM for Second order ODEs

We now extend our analysis to the second order ODE with constant coefficients given by

$$u''(x) + au'(x) + bu^n(x) = g(x), u(0) = c_0, u'(0) = c_1 \qquad ...(24)$$

Where $n \geq 1$

The MVIM admits the following for Eqn. (24)

$$u(x) = c_0 + xc_1 + \int_0^x (t-x)\left(u''(t) + au'(t) + bu^n(t) - g(t)\right)dt \qquad ...(25)$$

$$u_0(x) = c_0 + xc_1 \qquad ...(26)$$

$$u_1(x) = \int_0^x (t-x)\left(u_0''(t) + au_0'(t) + bu_0{}^n(t) - g(t)\right)dt \qquad ...(27)$$

$\vdots$

$$u_{n+1}(x) = \int_0^x (t-x) \sum_{m=0}^{n} \left(u_m''(t) + au_m'(t) + bu_m^n(t) - g(t)\right)dt - \sum_{m=0}^{n-1} u_m(x) \qquad ...(28)$$

For $n \geq 1$

Recall that

$$u(x) = u_0(x) + \sum_{m=1}^{n-1} u_m(x) \qquad ...(29)$$

2.6 MVIM for Third order ODEs

We now consider the third order linear ODE with constant coefficients given by

$$u'''(x) + au''(x) + bu'(x) + cu^n(x) = g(x), u(0) = c_0, u'(0) = c_1, u''(0) = c_2 \qquad ...(30)$$

Where $n \geq 1$

The MVIM admits the following for Eqn. (30)

$$u(x) = c_0 + xc_1 + \frac{x^2}{2!}c_2 - \frac{1}{2!}\int_0^x (t-x)^2 \left(u'''(t) + au''(t) + bu'(t) + cu^n(t) - g(t)\right)dt \qquad ...(31)$$

$$u_0(x) = c_0 + xc_1 + \frac{x^2}{2!}c_2 \qquad ...(32)$$

$$u_1(x) = -\frac{1}{2!}\int_0^x (t-x)^2 \left(u_0'''(t) + au_0''(t) + bu_0'(t) + cu_0^n(t) - g(t)\right)dt \qquad ...(33)$$

$$\vdots$$

$$u_{n+1}(x) = -\frac{1}{2!}\int_0^x (t-x)^2 \sum_{m=0}^{n}\left(u_m'''(t) + au_m''(t) + bu_m'(t) + cu_m^n(t) - g(t)\right)dt - \sum_{m=0}^{n-1} u_m(x) \qquad ...(34)$$

For $n \geq 1$

Recall that

$$u(x) = u_0(x) + \sum_{m=1}^{n-1} u_m(x) \qquad ...(35)$$

3. NUMERICAL RESULTS

Example 1:

Consider the first order homogeneous ODE (Odibat, 2006) and (A. Wazwaz, 2007b)

$$u'(x) - 2xu(x) = 0\ ,\ u(0) = 1 \qquad ...(36)$$

With exact solution

$$u(x) = e^{x^2}$$

In view of **(2.2)**, The MVIM admits the following for Eqn. (36)

$$u(x) = 1 - \int_0^x (u'(t) - 2\,tu(t))\,dt \qquad ...(37)$$

$$u_0(x) = 1$$

$$u_1(x) = x^2$$

$$u_2(x) = \frac{1}{2}x^4$$

$$u_3(x) = \frac{1}{6}x^6$$

$$u_4(x) = \frac{1}{24}x^8$$

$$u_5(x) = \frac{1}{120}x^{10}$$

$$u_6(x) = \frac{1}{720}x^{12}$$

$$\vdots$$

$$u(x) = 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \frac{1}{24}x^8 + \frac{1}{120}x^{10} + \frac{1}{720}x^{12} + \ldots \qquad \ldots(38)$$

Table 1: Numerical Results of Example 1

x	EXACT	MVIM	ERROR
0	1	1	0
0.1	1.01005	1.01005	2.22E-16
0.2	1.040811	1.040811	3.29E-14
0.3	1.094174	1.094174	9.6E-12
0.4	1.173511	1.173511	5.43E-10
0.5	1.284025	1.284025	1.25E-08
0.6	1.433329	1.433329	1.63E-07
0.7	1.632316	1.632315	1.43E-06
0.8	1.896481	1.896471	9.48E-06
0.9	2.247908	2.247858	5.04E-05
1	2.718282	2.718056	0.000226

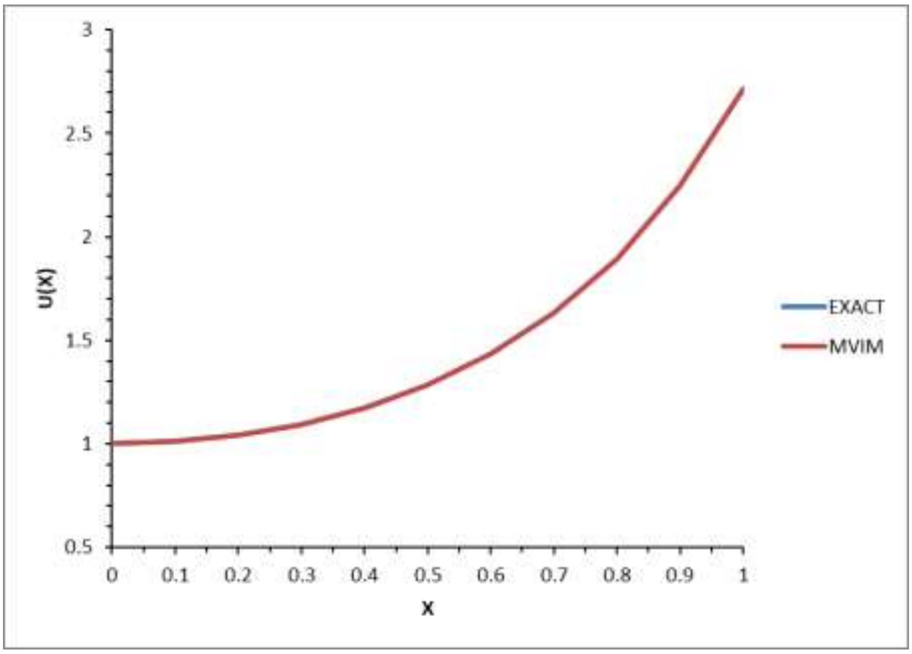

Figure 1: Comparison of approximate solutions obtained by the MVIM and the exact solution

Table 2: Comparison of Numerical Results of Example 1

x	EXACT	MVIM (n=6)	VIM (n=12)	ADM (n=5)
0	1	1	1	1
0.1	1.01005	1.01005017	1.01005017	1.01005017
0.2	1.040811	1.04081077	1.04081077	1.04081077
0.3	1.094174	1.09417428	1.09417428	1.09417423

0.4	1.173511	1.17351087	1.17351085	1.17350997
0.5	1.284025	1.2840254	1.28402507	1.28401693
0.6	1.433329	1.43332925	1.43332623	1.43327584
0.7	1.632316	1.63231479	1.63229556	1.63206017
0.8	1.896481	1.8964714	1.89637596	1.89548117
0.9	2.247908	2.24785755	2.24746529	2.24455963
1	2.718282	2.71805556	2.71666667	2.70833333

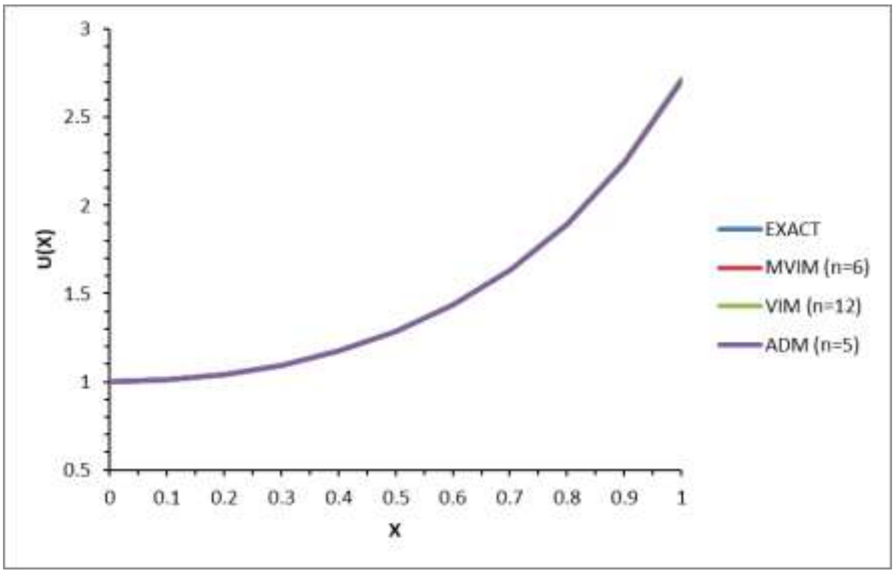

Figure 2: Comparison of approximate solutions obtained by the MVIM, VIM, ADM and the exact solution

Example 2:

Consider the first order non-homogeneous ODE (Wazwaz, 2009):

$$u'(x) - u(x) = e^{x},\ u(0) = 0. \qquad ...(39)$$

With exact solution

$$u(x) = xe^{x}$$

In view of **(2.1)**, The MVIM admits the following for Eqn. (39)

$$u(x) = 0 - \int_0^x \left(u'(t) - u(t) - e^{t}\right) dt$$

$$u_0(x) = 0$$

$$u_1(x) = -1 + e^{x}$$

$$u_2(x) = -1 - x + e^{x}$$

$$u_3(x) = -1 - x + e^{x} - \frac{1}{2}x^2$$

$$u_4(x) = -1 - x + e^{x} - \frac{1}{2}x^2 - \frac{1}{6}x^3$$

$$\vdots$$

$$u(x) = -4 + 4e^{x} - 3x - x^2 - \frac{1}{6}x^3 + ... \qquad ...(40)$$

Table 3: Numerical Results of Example 2

x	EXACT	MVIM	ERROR
0	0	0	0
0.1	0.110517	0.110517	8.62E-08
0.2	0.244281	0.244278	2.85E-06
0.3	0.404958	0.404935	2.24E-05
0.4	0.59673	0.596632	9.78E-05
0.5	0.824361	0.824052	0.000309
0.6	1.093271	1.092475	0.000796
0.7	1.409627	1.407844	0.001783
0.8	1.780433	1.77683	0.003602
0.9	2.213643	2.206912	0.00673
1	2.718282	2.706461	0.011821

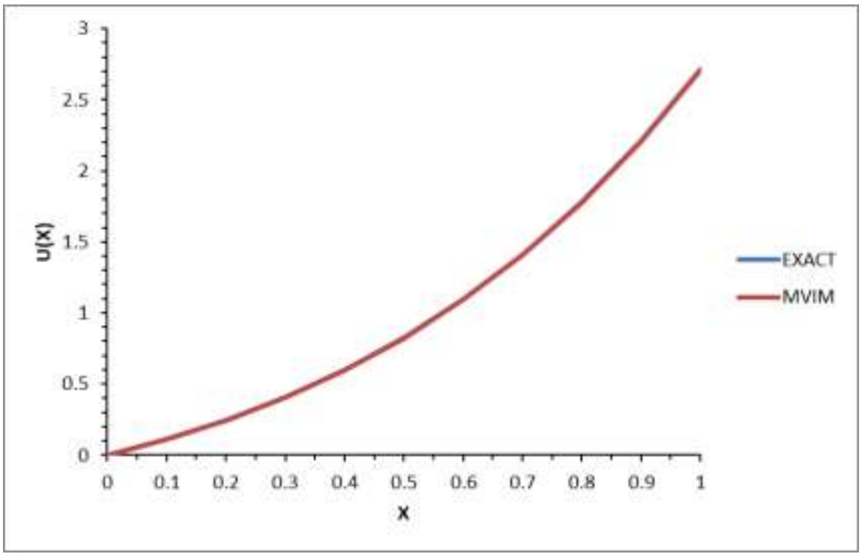

Figure 3: Comparison of approximate solutions obtained by the MVIM and the exact solution for example 2

Example 3:

Consider the nonlinear first order initial value problem (Ogunrinde, 2019):

$$u'(x) = u^2(x), u(0) = 1, \qquad ...(41)$$

With exact solution

$$u(x) = \frac{1}{1-x}$$

In view of **(2.1)**, The MVIM admits the following for Eqn. (41)

$$u(x) = 1 - \int_0^x (x-t)^{1-1} \cdot \left(\frac{d}{dt} u(t) - u(t)^2 \right) dt \qquad ...(42)$$

$$u_0(x) = 1$$

$$u_1(x) = x$$

$$u_2(x) = \frac{1}{3}x^3 + x^2$$

$$u_3(x) = \frac{1}{63}x^7 + \frac{1}{9}x^6 + \frac{1}{3}x^5 + \frac{2}{3}x^4 + \frac{2}{3}x^3$$

$$u_4(x) = \frac{1}{59535}x^{15} + \frac{1}{3969}x^{14} + \frac{1}{567}x^{13} + \frac{1}{126}x^{12} + \frac{5}{189}x^{11} + \frac{22}{315}x^{10} + \frac{86}{567}x^9 + \frac{71}{252}x^8 + \frac{4}{9}x^7 + \frac{5}{9}x^6 + \frac{8}{15}x^5 + \frac{1}{3}x^4$$

$$\vdots$$

$$u(x) = 1 + x + \frac{1}{3}x^3 + x^2 + \frac{1}{63}x^7 + \frac{1}{9}x^6 + \frac{1}{3}x^5 + \frac{2}{3}x^4 + \frac{2}{3}x^3 + \frac{1}{59535}x^{15} + \frac{1}{3969}x^{14} + \frac{1}{567}x^{13} + \frac{1}{126}x^{12} + \frac{5}{189}x^{11} + \frac{22}{315}x^{10} + \frac{86}{567}x^9 + \frac{71}{252}x^8 + \frac{4}{9}x^7 + \frac{5}{9}x^6 + \frac{8}{15}x^5 + \frac{1}{3}x^4$$

...(43)

Table 4: Numerical Results of Example 3

x	EXACT	MVIM	Absolute Error
0	1	1	0
0.1	1.111111	1.111109	1.72877E-06
0.2	1.25	1.249927	7.33013E-05
0.3	1.428571	1.427815	0.000756823
0.4	1.666667	1.662193	0.004474165
0.5	2	1.980076	0.019923668
0.6	2.5	2.423781	0.0762192

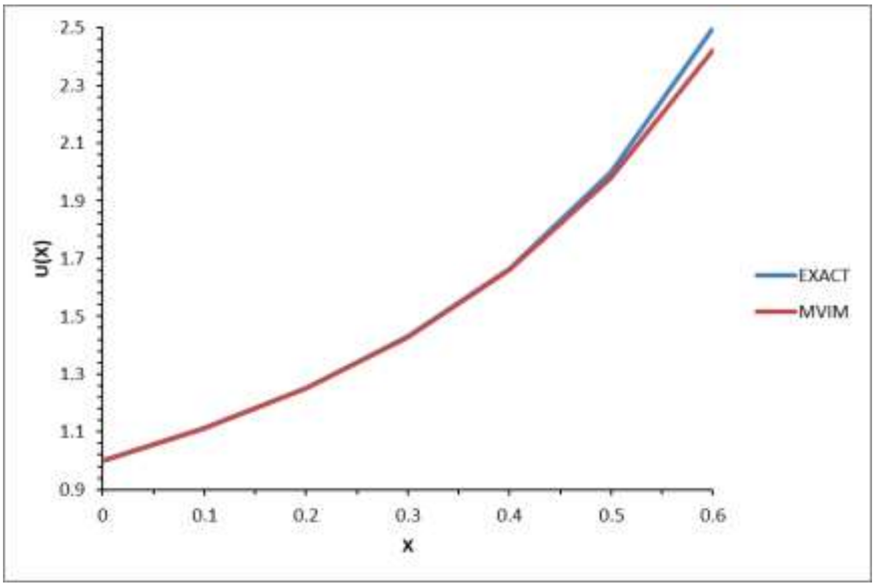

Figure 4: Comparison of approximate solutions obtained by the MVIM and the exact solution for example 3

Example 4:

Let us consider the nonlinear initial value problem (Hemeda, 2015):

$$u'(x)+u^2(x)=1, u(0)=0, \qquad ...(44)$$

With exact solution

$$u(x)=\tanh(x)$$

In view of **(2.1)**, The MVIM admits the following for Eqn. (44)

$$u(x)=0-\int_0^x (t-x)^{1-1}\cdot\left(\frac{d}{dt}u(t)+u^2(t)-1\right)dt \qquad ...(45)$$

$$u_0(x)=0$$

$$u_2(x)=-\frac{1}{3}x^3$$

$$u_3(x)=-\frac{1}{63}x^7+\frac{2}{15}x^5$$

$$u_4(x)=-\frac{1}{59535}x^{15}+\frac{4}{12285}x^{13}-\frac{134}{51975}x^{11}+\frac{38}{2835}x^9-\frac{4}{105}x^7$$

$$u_5(x)=-\frac{1}{109876902975}x^{31}+\frac{8}{21210236775}x^{29}-\frac{100732}{14119435204875}x^{27}+\frac{256948}{3016973334375}x^{25}-\frac{12676238}{16962094524375}x^{23}+\frac{29756}{5746615875}x^{21}-\frac{24022}{820945125}x^{19}+\frac{1522814}{10854718875}x^{17}-\frac{366292}{638512875}x^{15}+\frac{11344}{6081075}x^{13}-\frac{148}{31185}x^{11}+\frac{8}{945}x^9$$

$$\vdots$$

$$u(x)=x-\frac{1}{3}x^3-\frac{17}{315}x^7+\frac{2}{15}x^5-\frac{377017}{638512875}x^{15}+\frac{13324}{6081075}x^{13}-\frac{1142}{155925}x^{11}+\frac{62}{2835}x^9-\frac{1}{109876902975}x^{31}+\frac{8}{21210236775}x^{29}-\frac{100732}{14119435204875}x^{27}+\frac{256948}{3016973334375}x^{25}-\frac{12676238}{16962094524375}x^{23}+\frac{29756}{5746615875}x^{21}-\frac{24022}{820945125}x^{19}+\frac{1522814}{10854718875}x^{17} \qquad ...(46)$$

Table 5: Numerical Results of Example 4

x	EXACT	MVIM	ERROR
0	0	0	0
0.1	0.099668	0.099668	1.52E-14
0.2	0.197375	0.197375	3.04E-11
0.3	0.291313	0.291313	2.52E-09
0.4	0.379949	0.379949	5.6E-08
0.5	0.462117	0.462118	6.04E-07
0.6	0.53705	0.537054	4.1E-06
0.7	0.604368	0.604388	2.01E-05

0.8	0.664037	0.664115	7.79E-05
0.9	0.716298	0.716549	0.000251
1	0.761594	0.762293	0.000699

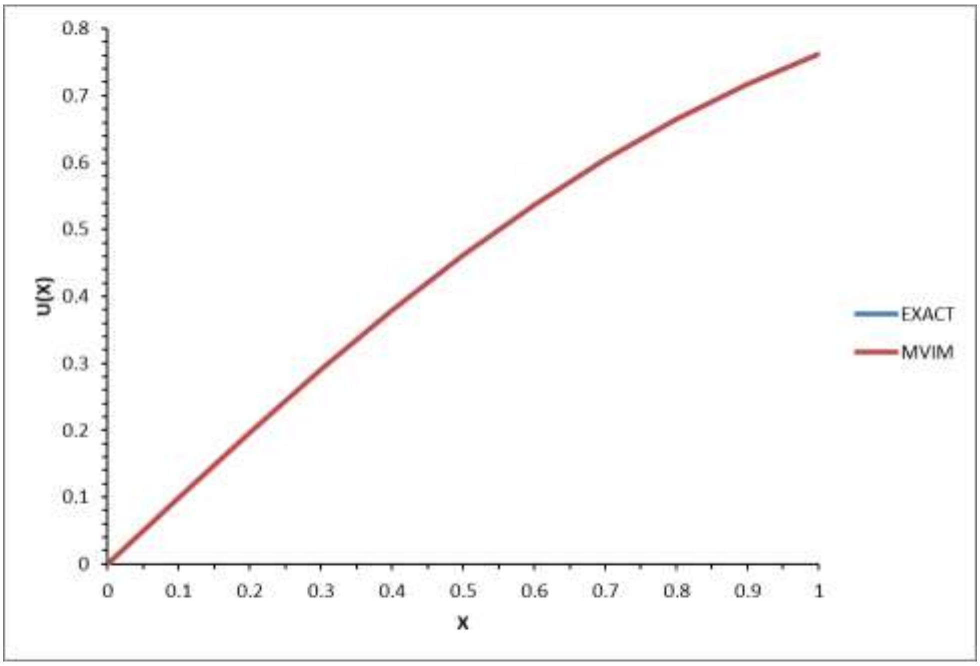

Figure 5: Comparison of approximate solutions obtained by the MVIM and the exact solution for Example 4

Table 6: Comparison of Numerical Results of Example 4

x	EXACT	MVIM	NIM	MVIM Error	NIM Error
0	0	0	0	0	0
0.1	0.099668	0.099668	0.099668	1.51684E-14	1.5168E-14
0.2	0.197375	0.197375	0.197375	3.04029E-11	3.0403E-11
0.3	0.291313	0.291313	0.291313	2.51513E-09	2.515E-09
0.4	0.379949	0.379949	0.379949	5.60137E-08	5.599E-08
0.5	0.462117	0.462118	0.462118	6.03873E-07	6.0286E-07
0.6	0.53705	0.537054	0.537054	4.09609E-06	4.074E-06
0.7	0.604368	0.604388	0.604388	2.01192E-05	1.9823E-05
0.8	0.664037	0.664115	0.664112	7.78707E-05	7.509E-05
0.9	0.716298	0.716549	0.716529	0.000250927	0.00023098
1	0.761594	0.762293	0.762178	0.000699182	0.00058364

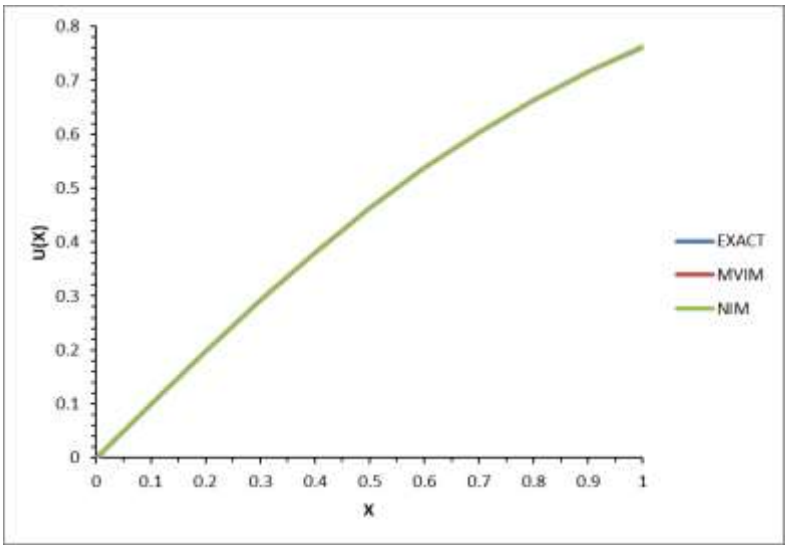

Figure 6: Comparison of approximate solutions obtained by the MVIM, NIM, and the exact solution for Example 4

3.1 Discussion of Results

In work, we applied the modified variational iteration method to ODEs of distinct orders and with variable and constant coefficients. We presented the analysis of several cases beginning from the first-order through the n^{th} -order linear and nonlinear differential equations, inclusively. The illustrative examples, that we examined, show that the MVIM is reliable and efficient over existing methods. It is clearly seen that the method gives rapid convergent successive approximations without any restrictive assumptions or transformation that may change the physical behavior of the problem. For nonlinear equations that arise frequently to express nonlinear phenomenon, the MVIM facilitates the computational work and gives the solution rapidly if compared when compared to other methods. For nonlinear inhomogeneous problems, the MVIM a few number of iterations are require.

3.2 Conclusion

The aim of this work has been to construct an approximate solution of linear and nonlinear differential equations. The aim and objectives has been achieved by using the modified variational iteration method. The modification is used in a direct manner without using linearization or Adomian polynomials, perturbation or restrictive assumptions. There are four important points to make here. First, the MVIM provide the solutions in terms of convergent series with easily computable components. Secondly, the high agreement of the approximation of $u(x)$ between the MVIM used in this work is clear and remarkable. So, they can be used as an alternative methods other equations. Third, the modified variational iteration method is more effective and overcome the difficulty arising in calculating the Lagrange's multiplier and the Adomian polynomials. Finally, the choice of the initial approximation or the trial function, which can be freely selected by imposing the initial conditions, in the variational iteration method may leading to approximation of $u(x)$ which converges rapidly and faster than those obtained from the other methods.

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