International Journal of Research Publication and Reviews

Journal homepage: www.ijrpr.com ISSN 2582-7421

On the Ternary Non-Homogeneous Quintic Equation

$$x^2 + k y^2 = (p^2 + k q^2)^n z^5$$

S. Vidhyalakshmi[1], M. A. Gopalan[2]

[1]Assistant Professor, Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Trichy-620 002, Tamil Nadu, India.

[2]Professor, Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Trichy-620 002, Tamil Nadu, India.

ABSTRACT:

The ternary non-homogeneous quantic equation given by $x^2 + k y^2 = (p^2 + k q^2)^n z^5$, where p, q are non-zero integers and k is a non-zero positive square-free integer, is analysed for determining its distinct integer solutions. Also, a generation formula for the integer solutions to the given quintic equation, being given its particular solution, is illustrated.

Key words: Ternary quintic, Non-homogeneous quintic, Integer solutions

Introduction:

It is well-known that the Diophantine equations, homogeneous or non-homogeneous, have aroused the interest of many mathematicians. In particular, one may refer [1-9] for quintic equations with three unknowns. The above problems motivated us to search for the distinct integer solutions to ternary non-homogeneous quintic equation

$$x^2 + k y^2 = (p^2 + k q^2)^n z^5$$

Also, a general formula for generating sequence of integer solutions to the considered quintic equation being given its particular solution is illustrated.

Method of analysis:

The ternary non-homogeneous quintic equation under consideration is

$$x^2 + k y^2 = (p^2 + k q^2)^n z^5, \tag{1}$$

where p,q are non-zero integers and k is a non-zero positive square-free integer.

The process of determining non-zero distinct integer solutions to (1) is illustrated below:

Assume

$$z = z(a,b,k) = a^2 + k b^2 \tag{2}$$

Taking $n = 0$ in (1) , it is written as

$$x^2 + k y^2 = z^5 \tag{3}$$

Using (2) in (3) and employing the method of factorization, consider

$$x + i\sqrt{k}y = (a + i\sqrt{k}b)^5$$

Equating the rational and irrational parts, it is seen that

$$x = a^5 - 10k a^3 b^2 + 5k^2 a b^4, y = 5a^4 b - 10k a^2 b^3 + k^2 b^5 \quad (4)$$

Thus, (2) and (4) represent the integer solutions to (3). For convenience , denote the values of x, y given by (4) by the notations x (a,b,k;0), y(a,b,k;0) respectively.

Taking n=1 in (1), it is written as

$$x^2 + k y^2 = (p^2 + k q^2) z^5 \quad (5)$$

Following the procedure as presented above, the integer solutions to (5) are given by

$$x = x(a,b,k;1) = p\,x(a,b,k;0) - k q\, y(a,b,k;0),$$
$$y = y(a,b,k;1) = p\,y(a,b,k;0) + q\,x(a,b,k;0)$$

along with (2). In general, after performing some algebra,the corresponding integer solutions to (1) are given by

$$\left.\begin{aligned} x &= x(a,b,k;n) = \frac{\alpha^n + \beta^n}{2} x(a,b,k;0) + \frac{i\sqrt{k}(\alpha^n - \beta^n)}{2} y(a,b,k;0), \\ y &= y(a,b,k;n) = \frac{(-i)(\alpha^n - \beta^n)}{2\sqrt{k}} x(a,b,k;0) + \frac{\alpha^n + \beta^n}{2} y(a,b,k;0), \\ z &= z(a,b,k) = a^2 + k b^2 \end{aligned}\right\} \quad (6)$$

where $\alpha = p + i\sqrt{k}q, \beta = p - i\sqrt{k}q,$

A few examples are given in Table 1 below:

n	$x(1,1,2;n)$	$y(1,1,2;n)$	$z(1,1,2)$
0	1	-11	3
1	$p + 22q$	$q - 11p$	3
2	$p^2 + 44pq - 2q^2$	$-11p^2 + 2pq + 22q^2$	3

The recurrence relations satisfied by the values of x, y given by (6) are presented below:

$$x(a,b,k;n+2) - 2p\,x(a,b,k;n+1) + (p^2 + k q^2)\,x(a,b,k;n) = 0,$$
$$y(a,b,k;n+2) - 2p\,y(a,b,k;n+1) + (p^2 + k q^2)\,y(a,b,k;n) = 0, n = 0,1,2,\ldots$$

Note:1

It is to be noted that (3) is also satisfied by

$$x = m(m^2 + kn^2)^{5s-3}, y = n(m^2 + kn^2)^{5s-3}, z = (m^2 + kn^2)^{2s-1}$$

Note:2

There is yet another solution to (3) as presented below:

Write (3) as

$$x^2 + k y^2 = z^5 * 1 \quad (7)$$

The integer 1 on the R.H.S. of (7) is expressed as

$$1=\frac{(kr^2-s^2+i\sqrt{k}2rs)(kr^2-s^2-i\sqrt{k}2rs)}{(kr^2+s^2)^2} \qquad (8)$$

Substituting (2) & (8) in (7) and applying the method of factorization , consider

$$x+i\sqrt{k}y=\frac{(kr^2-s^2+i\sqrt{k}2rs)}{(kr^2+s^2)}(a+i\sqrt{k}b)^5 \qquad (9)$$

Equating the real and imaginary parts in (9) , the values of x, y are obtained.

Replacing a, b by $(kr^2+s^2)A, (kr^2+s^2)B$ respectively in the above resulting values of x, y and (2) , the corresponding integer solutions to (3) are obtained.

Observation: Generation formula

Let $(x_0(a,b,k;n), y_0(a,b,k;n), z_0(a,b,k))$ be a particular solution to (1).

Then, the formula for generating a sequence of integer solutions to (1) is presented below:

$$x_s=\frac{\alpha^s+k\beta^s}{1+k}x_0(a,b,k;n)+\frac{k(\alpha^s-\beta^s)}{1+k}y_0(a,b,k;n),$$
$$y_s=\frac{\alpha^s-\beta^s}{1+k}x_0(a,b,k;n)+\frac{k\alpha^s+\beta^s}{1+k}y_0(a,b,k;n),$$
$$z_s=(1+k)^{2s}z_0(a,b,k) \quad ,s=1,2,3,...$$

where

$$\alpha=(1+k)^5 \quad ,\beta=-(1+k)^5$$

An example has been given in Table 2 below:

Table 2- Example

n	s	$x_s(1,1,2;n)$	$y_s(1,1,2;n)$	$z_s(1,1,2)$
0	0	1	-11	3
	1	-45*81	-9*81	27
	2	243^2	$-11*243^2$	243

Conclusion:

In this paper, an attempt has been made to obtain non-zero distinct integer solutions to the ternary non-homogeneous quintic equation

$$x^2+ky^2 = (p^2+kq^2)^n z^5$$

As the quintic equations are rich in variety,one may search for the integer solutions to other choices of quintic equations with three or more unknowns.

References

1. M.A.Gopalan and Sangeetha.G ,Integral solutions of ternary quintic Diophantine equation $x^2+y^2=(k^2+1)z^5$, Bulletin of pure and applied sciences. Vol.29 No:1,23-28,(2010).
2. M.A.Gopalan ,A.VijayaSankar,An Interesting Diophantine Problem $x^3-y^3=2z^5$ Advances in Mathematics,Scientific Developments and Engineering Applications,Narosa Publishing House,1-6,2010
3. M. A. Gopalan, A.VijayaSankar, Integral solutions of ternary quintic Diophantine equation, $x^2+(2k+1)y^2=z^5$ International journal of Mathematical sciences.,Vol.19(1-2), 165-169, (2010).
4. S. Vidhyalakshmi., K. Lakshmi., M. A. Gopalan,, Integral solutions of non-homogeneous ternary quintic Diophantine equation $ax^2+by^2=(a+b)z^5$, $a>b>0$, Archimedes journal of Mathematics ., vol.3, No:2 , 197-204 (2013).
5. M.A.Gopalan, S.Vidhyalakshmi, E. Premalatha.M. Manjula, M and N.Thiruniraiselvi, On the non-homogeneous ternary quintic equation, $2(x^2+y^2)-3xy=7^{2n}z^5$ Bessel J.Math., Vol.3(3), Pp.249-254, 2013

6. M.A.Gopalan,G.Sumathi and S.Vidhyalakshmi,Integral Solutions of the Non-Homogeneous. Ternary Quintic Equation Interms of Pell Sequence $x^3 + y^3 + xy(x + y) = 2z^5$,JAMS , 6(1) ,56-62 ,2013
7. M.A.Gopalan,G.Sumathi and S.Vidhyalakshmi,Integral Solutions of Non-Homogeneous Quintic Equation with Three Unknowns $x^2 + y^2 - xy + x + y + 1 = (k^2 + 3)^n z^5$,IJIRSET , 2(4) ,920-925 ,2013
8. S. Vidhyalakshmi, K. Lakshmi and M.A.Gopalan, Integral solutions of the non-homogeneous ternary quintic equation $ax^2 - by^2 = (a-b)z^5$, International Journal of Computational Engineering Research, 3(4), 45-50, 2013.
9. M.A.Gopalan, N.Thiruniraiselvi, R.Presenna, Quintic with three unknowns $3(x^2 + y^2) - 2xy + 2(x + y) + 1 = 33z^5$, International Journal of Multidisciplinary Research and Modern Engineering, Vol.1(1),Pp.171-173, 2015