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A Note on Eigenvalues of Bicomplex Matrix

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A B S T R A C T

In this paper, we have studied eigenvalues and eigenvectors of the bicomplex matrix and investigated their properties and established some results. We have also established some results on the eigenvalues of some special bicomplex matrices.

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1. Introduction

Throughout this paper, the set of Bicomplex numbers is denoted by $\mathbb{C}_2$ and the sets of complex and real numbers are denoted by $\mathbb{C}_1$ and $\mathbb{C}_0$,respectively. For detail of the theory (cf. **[1, 2, 3]**).

Definition 1.1 (Bicomplex numbers). The set of bicomplex numbers is defined as:

$$\mathbb{C}_2 = \{x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 : x_1, x_2, x_3, x_4 \in \mathbb{C}_0\}$$

where $i_1 \neq i_2, i_1^2 = i_2^2 = -1$ and, $i_1i_2 = i_2i_1$.

We shall use the notations $\mathbb{C}(i_1)$ and $\mathbb{C}(i_2)$ for the following sets:

$$\mathbb{C}(i_1) = \{x + i_1y : x, y \in \mathbb{C}_0\}$$

$$\mathbb{C}(i_2) = \{x + i_2y : x, y \in \mathbb{C}_0\}$$

Definition 1.2 (Hyperbolic numbers). The set of Hyperbolic Numbers is defined as

$$\mathbb{H} = \{x + i_1i_2y : x, y \in \mathbb{C}_0\}$$

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Definition 1.3 (Idempotent elements). Besides 0 and 1, there are exactly two non-trivial idempotent elements exist in $\mathbb{C}_2$, denoted as e_1 and e_2 and defined as $e_1 = \frac{1+i_1 i_2}{2}$ and $e_2 = \frac{1-i_1 i_2}{2}$. Note that $e_1 + e_2 = 1$ and $e_1 e_2 = e_2 e_1 = 0$.

Definition 1.4 (Cartesian idempotent set). [4] Cartesian idempotent set X determined by X_1 and X_2 is denoted as $X_1 \times_e X_2$ and is defined as

$$X = X_1 \times_e X_2 = \{\xi \in X : \xi = ae_1 + be_2, (a, b) \in X_1 \times X_2\}$$

(i) The Cartesian idempotent set $\mathbb{C}_2$ determined by $\mathbb{C}(i_1)$ is given as :

$$\mathbb{C}_2 = \mathbb{C}(i_1) \times_e \mathbb{C}(i_1) = \mathbb{C}(i_1)e_1 + \mathbb{C}(i_1)e_2$$
$$= \{\xi \in \mathbb{C}_2 : \xi = {}^1\xi e_1 + {}^2\xi e_2, ({}^1\xi, {}^2\xi) \in \mathbb{C}(i_1) \times \mathbb{C}(i_1)\}$$

(ii) The Cartesian idempotent set $\mathbb{C}_2$ determined by $\mathbb{C}(i_2)$ is given as :

$$\mathbb{C}_2 = \mathbb{C}(i_2) \times_e \mathbb{C}(i_2) = \mathbb{C}(i_2)e_1 + \mathbb{C}(i_2)e_2$$
$$= \{\xi \in \mathbb{C}_2 : \xi = \xi_1 e_1 + \xi_2 e_2, (\xi_1, \xi_2) \in \mathbb{C}(i_2) \times \mathbb{C}(i_2)\}$$

1.1 Idempotent Representation of Bicomplex Numbers.

The bicomplex numbers can be represented in two different idempotent forms w.r.t. the elements from $\mathbb{C}(i_1)$ and $\mathbb{C}(i_2)$, explained as follows :

(a) The $\mathbb{C}(i_1)$– idempotent representation of Bicomplex Numbers is given by

$$\xi = x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 = (x_1 + i_1x_2) + i_2(x_3 + i_1x_4)$$
$$= z_1 + i_2z_2 = (z_1 - i_1z_2)e_1 + (z_1 + i_1z_2)e_2 = {}^1\xi e_1 + {}^2\xi e_2,$$

where, ${}^1\xi = z_1 - i_1z_2 = (x_1 + x_4) + i_1(x_2 - x_3)$,

$${}^2\xi = z_1 + i_1z_2 = (x_1 - x_4) + i_1(x_2 + x_3) \in \mathbb{C}(i_1)$$

(b) The $\mathbb{C}(i_2)$–idempotent representation of Bicomplex Numbers is given by

$$\xi = x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 = (x_1 + i_2x_3) + i_1(x_2 + i_2x_4)$$
$$= w_1 + i_1w_2 = (w_1 - i_2w_2)e_1 + (w_1 + i_2w_2)e_2 = \xi_1 e_1 + \xi_2 e_2,$$

where, $\xi_1 = w_1 - i_2w_2 = (x_1 + x_4) - i_2(x_2 - x_3)$,

$$\xi_2 = w_1 + i_2w_2 = (x_1 - x_4) + i_2(x_2 + x_3) \in \mathbb{C}(i_2)$$

1.2 Singular Elements in $\mathbb{C}_2$:

Let $\xi,\eta \in \mathbb{C}_2$ such that $\xi\eta = \eta\xi = 1$, then η is said to be a multiplicative inverse of ξ. The invertible elements are also called non-singular elements. The set of all singular elements in $\mathbb{C}_2$ is denoted as $\mathbb{O}_2$ and $\mathbb{C}_2\backslash\mathbb{O}_2$ is the set of all non-singular elements in $\mathbb{C}_2$.

1.2.1 Singular Elements in $\mathbb{C}_2$. We are providing some conditions for the singularity of the bicomplex number as follows:

(i) $\xi = z_1 + i_2 z_2 = (z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2 = {}^1\xi e_1 + {}^2\xi e_2$ is singular if and only if

$z_1^2 + z_2^2 = 0$ or ($z_1 - i_1 z_2 = 0$ or $z_1 + i_1 z_2 = 0$) or (${}^1\xi = 0$ or ${}^2\xi = 0$)

(ii) $\xi = w_1 + i_1 w_2 = (w_1 - i_2 w_2)e_1 + (w_1 + i_2 w_2)e_2 = \xi_1 e_1 + \xi_2 e_2$ is singular if and only if

$w_1^2 + w_2^2 = 0$ or ($w_1 - i_2 w_2 = 0$ or $w_1 + i_2 w_2 = 0$) or ($\xi_1 = 0$ or $\xi_2 = 0$)

Definition 1.5 (Principal Ideals in $\mathbb{C}_2$). There are two principal ideals in $\mathbb{C}_2$ viz., $\mathbb{I}_1$ and $\mathbb{I}_2$ defined as

$$\mathbb{I}_1 = \{\xi e_1 : \xi = {}^1\xi e_1 + {}^2\xi e_2 \in \mathbb{C}_2\} = \{{}^1\xi e_1 : {}^1\xi \in \mathbb{C}(i_1)\}$$

$$\mathbb{I}_2 = \{\xi e_2 : \xi = {}^1\xi e_1 + {}^2\xi e_2 \in \mathbb{C}_2\} = \{{}^2\xi e_2 : {}^2\xi \in \mathbb{C}(i_1)\}$$

Note that $\mathbb{I}_1 \cap \mathbb{I}_2 = \{0\}$ and $\mathbb{I}_1 \cup \mathbb{I}_2 = \mathbb{O}_2$.

Definition 1.6 (Zero-divisors). Two non-zero bicomplex numbers ξ and η are said to be zero-divisors if $\xi\,\eta = 0$.

The relation $e_1 e_2 = e_2 e_1 = 0$ establishes the existence of zero divisors in $\mathbb{C}_2$.

Proposition 1.1 The bicomplex numbers ξ and η are divisors of zero if and only if $\xi \in \mathbb{I}_1\backslash\{0\}$ and $\eta \in \mathbb{I}_2\backslash\{0\}$.

Definition 1.7 (Norm on the Bicomplex space).

(i) Let $\xi = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 = z_1 + i_2 z_2 = {}^1\xi e_1 + {}^2\xi e_2 \in \mathbb{C}_2$. Then the norm in $\mathbb{C}_2$ is defined as

$$\|\xi\| = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2} = \sqrt{|z_1|^2 + |z_2|^2} = \sqrt{\frac{|{}^1\xi|^2 + |{}^2\xi|^2}{2}}$$

(ii) Let $\xi = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 = w_1 + i_1 w_2 = \xi_1 e_1 + \xi_2 e_2 \in \mathbb{C}_2$. Then the norm in $\mathbb{C}_2$ is defined as

$$\|\xi\| = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2} = \sqrt{|w_1|^2 + |w_2|^2} = \sqrt{\frac{|\xi_1|^2 + |\xi_2|^2}{2}}$$

Note that $\mathbb{C}_2$ becomes a modified Banach algebra, in the sense that for any $\xi, \eta \in \mathbb{C}_2$,

$$\| \xi \eta \| \leq \sqrt{2} \; \| \xi \| \, \| \eta \|.$$

1.3. Conjugations of Bicomplex Numbers.There are three different types of conjugations given on the bicomplex space and are explained as follows [3]:

(a) i_1-Conjugation

$$\xi^* = (z_1 + i_2 z_2)^* = \overline{z_1} + i_2 \overline{z_2}, \; \forall \; z_1, z_2 \in \mathbb{C}(i_1)$$
$$= ({}^1\xi e_1 + {}^2\xi e_2)^* = \overline{{}^2\xi}\, e_1 + \overline{{}^1\xi}\, e_2, \forall\, {}^1\xi, {}^2\xi \in \mathbb{C}(i_1)$$
$$= (w_1 + i_1 w_2)^* = w_1 - i_1 w_2, \forall \; w_1, w_2 \in \mathbb{C}(i_2)$$
$$= (\xi_1 e_1 + \xi_2 e_2)^* = \xi_2 e_1 + \xi_1 e_2, \forall \; \xi_1, \xi_2 \in \mathbb{C}(i_2)$$

(b) i_2-Conjugation

$$\xi^{\#} = (z_1 + i_2 z_2)^{\#} = z_1 - i_2 z_2, \forall \; z_1, z_2 \in \mathbb{C}(i_1)$$
$$= ({}^1\xi e_1 + {}^2\xi e_2)^{\#} = {}^2\xi e_1 + {}^1\xi e_2, \forall\, {}^1\xi, {}^2\xi \in \mathbb{C}(i_1)$$
$$= (w_1 + i_1 w_2)^{\#} = \overline{w_1} \; + i_1 \overline{w_2} \;, \forall\, w_1, w_2 \in \mathbb{C}(i_2)$$
$$= (\xi_1 e_1 + \xi_2 e_2)^{\#} = \overline{\xi_2}\, e_1 + \overline{\xi_1}\, e_2, \forall \; \xi_1, \xi_2 \in \mathbb{C}(i_2).$$

(c) $i_1 i_2$-Conjugation

$$\xi' = (z_1 + i_2 z_2)' = \overline{z_1} \; - i_2 \overline{z_2}, \; \forall \; z_1, z_2 \in \mathbb{C}(i_1)$$
$$= ({}^1\xi e_1 + {}^2\xi e_2)' = \overline{{}^1\xi}\, e_1 + \overline{{}^2\xi}\, e_2, \forall\, {}^1\xi, {}^2\xi \in \mathbb{C}(i_1)$$
$$= (w_1 + i_1 w_2)' = \overline{w_1} \; - i_1 \overline{w_2}, \; \forall \; w_1, w_2 \in \mathbb{C}(i_2)$$
$$= (\xi_1 e_1 + \xi_2 e_2)' = \overline{\xi_1} e_1 + \overline{\xi_2} e_2, \forall \; \xi_1, \xi_2 \in \mathbb{C}(i_2)$$

2.Bicomplex Matrix

In this section, we discussed the bicomplex matrices along with their properties and some results. We established the results concerning the conjugations of the bicomplex matrices **[5, 6, 7, 8]**.

Here, we denote $\mathbb{C}_2^{m \times n} = \{A = [\xi_{ij}]_{m \times n} : \xi_{ij} \in \mathbb{C}_2\}$ as the set of $m \times n$ matrices with bicomplex entries. Let $A = [\xi_{ij}]_{m \times n} \in \mathbb{C}_2^{m \times n} \Rightarrow \xi_{ij} \in \mathbb{C}_2$

2.1.Idempotent Representation of Bicomplex Matrix. We can represent the Bicomplex Matrix in two different idempotent forms in terms of $\mathbb{C}^{m \times n}(i_1)$ and $\mathbb{C}^{m \times n}(i_2)$, explained as follows :

(i) The $\mathbb{C}^{m \times n}(i_1)$-idempotent representation of Bicomplex Matrix $A = [\xi_{ij}]_{m \times n} \in \mathbb{C}_2^{m \times n}$ is given by

$$A = [\xi_{ij}]_{m \times n} = [{}^1\xi_{ij}]_{m \times n} e_1 + [{}^2\xi_{ij}]_{m \times n} e_2 = {}^1A e_1 + {}^2A e_2,$$

where ${}^{1}A = [{}^{1}\xi_{ij}]_{m\times n}, {}^{2}A = [{}^{2}\xi_{ij}]_{m\times n} \in \mathbb{C}^{m\times n}(i_1)$

(ii) The $\mathbb{C}^{m\times n}(i_2)$-idempotent representation of Bicomplex Matrix $A = [\xi_{ij}]_{m\times n} \in \mathbb{C}_2^{m\times n}$ is given by

$$A = [\xi_{ij}]_{m\times n} = [\xi_{1,ij}]_{m\times n}e_1 + [\xi_{2,ij}]_{m\times n}e_2 = A_1e_1 + A_2e_2,$$

where $A_1 = [\xi_{1,ij}]_{m\times n}, A_2 = [\xi_{2,ij}]_{m\times n} \in \mathbb{C}^{m\times n}(i_2)$

2.2. Determinant of Bicomplex Matrices. As, only square matrices can have determinant, so let $\mathbf{A = [\xi_{ij}]_{n\times n} \in \mathbb{C}_2^{n\times n}}$***. Then determinant of A is denoted as*** $\mathbf{det(A)}$***.***

Theorem 2.1. Let $A = [\xi_{ij}]_{n\times n} \in \mathbb{C}_2^{n\times n}$, then its determinant is given by

$$\det(A) = \det({}^{1}A)e_1 + \det({}^{2}A)e_2 \text{ or } \det(A) = \det(A_1)e_1 + \det(A_2)e_2.$$

Corollary 2.2. For $A, B \in \mathbb{C}_2^{n\times n}$, $\det(AB) = \det(A)\det(B)$.

2.3. Algebraic Structure of Bicomplex Matrices. Bicomplex Matrices have the following algebraic structures.

$\mathbb{C}_2^{m\times n}(\mathbb{C}_0), \mathbb{C}_2^{m\times n}(\mathbb{C}(i_1))$ and $\mathbb{C}_2^{m\times n}(\mathbb{C}(i_2))$ are linear space; $\mathbb{C}_2^{m\times n}(\mathbb{C}_2)$ is a $\mathbb{C}_2$ -module; $\mathbb{C}_2^{n\times n}(\mathbb{C}_0), \mathbb{C}_2^{n\times n}(\mathbb{C}(i_1))$ and $\mathbb{C}_2^{n\times n}(\mathbb{C}(i_2))$ are algebra.

Theorem 2.3. $A = {}^{1}Ae_1 + {}^{2}Ae_2 = A_1e_1 + A_2e_2 \in \mathbb{C}_2^{n\times n}$ is invertible iff ${}^{1}A$ and ${}^{2}A$ are invertible in $\mathbb{C}^{n\times n}(i_1)$ and A_1 and A_2 are invertible in $\mathbb{C}^{n\times n}(i_2)$.

Corollary 2.4. $A \in \mathbb{C}_2^{n\times n}$ is invertible iff $\det(A) \notin \mathbb{O}_2$.

2.4. Non-Singular and Singular Bicomplex Matrix. A matrix $\mathbf{A = [\xi_{ij}]_{n\times n} \in \mathbb{C}_2^{n\times n}}$ ***is said to be Non-Singular (Invertible) if*** $\mathbf{det(A) \notin \mathbb{O}_2}$***. It is said to be Singular (Non-invertible) matrix if*** $\mathbf{det(A) \in \mathbb{O}_2}$***.***

2.5. Conjugation of Bicomplex Matrix. Here, we have given three different types of conjugations of the bicomplex matrices. Let $\mathbf{A} = [\boldsymbol{\xi}]_{\mathbf{m\times n}} \in \mathbb{C}_{\mathbf{2}}^{\mathbf{n\times n}}$ ***be a bicomplex matrix. Then, its conjugations are defined as follows:***

(a) i_1-Conjugation

$A^* = ({}^1A\, e_1 + {}^2A\, e_2)^* = \overline{{}^2A}\, e_1 + \overline{{}^1A}\, e_2$ when ${}^1A, {}^2A \in \mathbb{C}^{m\times n}(i_1)$,

$A^* = (A_1 e_1 + A_2 e_2)^* = A_2 e_1 + A_1 e_2$ when $A_1, A_2 \in \mathbb{C}^{m\times n}(i_2)$.

(b) i_2-Conjugation

$A^{\#} = ({}^1Ae_1 + {}^2Ae_2)^{\#} = {}^2Ae_1 + {}^1Ae_2$ when ${}^1A, {}^2A \in \mathbb{C}^{m\times n}(i_1)$,

$A^{\#} = (A_1 e_1 + A_2 e_2)^{\#} = \overline{A}_2 e_1 + \overline{A}_1 e_2$ when $A_1, A_2 \in \mathbb{C}^{m\times n}(i_2)$.

(c) $i_1 i_2$-Conjugation

$A' = ({}^1A\, e_1 + {}^2A\, e_2)' = \overline{{}^1A}\, e_1 + \overline{{}^2A}\, e_2$ when ${}^1A, {}^2A \in \mathbb{C}^{m\times n}(i_1)$,

$A' = (A_1 e_1 + A_2 e_2)' = \overline{A}_1 e_1 + \overline{A}_2 e_2$ when $A_1, A_2 \in \mathbb{C}^{m\times n}(i_2)$.

Definition 2.1 (Hyperbolic Matrix) The set of Hyperbolic Matrix is defined as :

$$\mathbb{H}^{m\times n} = \{A = [\xi_{ij}]_{m\times n} : \xi_{ij} \in \mathbb{H}\}. \text{ If } A = {}^1Ae_1 + {}^2Ae_2 \in \mathbb{H}^{m\times n}, \text{ then } {}^1A, {}^2A \in \mathbb{C}_0^{m\times n}.$$

Definition 2.2 (Symmetric Matrix). A Bicomplex matrix $A \in \mathbb{C}_2^{n\times n}$ is said to be symmetric matrix if $A^t = A$.

Definition 2.3 (Skew-Symmetric Matrix). A Bicomplex matrix $A \in \mathbb{C}_2^{n\times n}$ is said to be Skew-Symmetric matrix if $A^t = -A$.

Definition 2.4 (Bicomplex Idempotent Matrix). A Bicomplex matrix $A \in \mathbb{C}_2^{n\times n}$ is said to be idempotent matrix if $A^2 = A$.

Definition 2.5 (Bicomplex Skew-Idempotent Matrix). A Bicomplex matrix $A \in \mathbb{C}_2^{n\times n}$ is said to be Skew-idempotent matrix if $A^2 = -A$.

Definition 2.6 (Bicomplex Involutory Matrix). A Bicomplex matrix $A \in \mathbb{C}_2^{n\times n}$ is said to be involutory matrix if $A^2 = I$.

Definition 2.7 (Bicomplex Skew-Involutory Matrix). A Bicomplex matrix $A \in \mathbb{C}_2^{n\times n}$ is said to be Skew- involutory matrix if $A^2 = -I$.

Definition 2.8 (Bicomplex Hermitian Matrix). There are three types of Bicomplex Hermitian Matrix, defined as follows:

(i) i_1-Hermitian Matrix

A Bicomplex matrix $A \in \mathbb{C}_2^{n\times n}$ is said to be i_1 -Hermitian matrix if

$(A^t)^* = (A^*)^t = A$.

(ii) i_2-Hermitian Matrix

A Bicomplex matrix $A \in \mathbb{C}_2^{n\times n}$ is said to be i_2 -Hermitian matrix if

$(A^t)^\# = (A^\#)^t = A.$

(iii) $i_1 i_2$-Hermitian Matrix

A Bicomplex matrix $A \in \mathbb{C}_2^{n\times n}$ is said to be $i_1 i_2$ -Hermitian matrix if

$(A^t)' = (A')^t = A.$

Note 2.1 There are some observations as given follows:

(a) If A is i_1-Hermitian Matrix then $A^* = A^t$.

(b) If A is i_2-Hermitian matrix then $A^\# = A^t$.

(c) If A is $i_1 i_2$-Hermitian matrix then $A' = A^t$.

Definition 2.9 (Bicomplex Skew- Hermitian Matrix). There are three types of bicomplex skew-Hermitian matrix, defined as follows:

(i) i_1-Skew-Hermitian Matrix

A Bicomplex matrix $A \in \mathbb{C}_2^{n\times n}$ is said to be i_1 -skew Hermitian matrix if

$(A^t)^* = (A^*)^t = -A$

(ii) i_2-skew-Hermitian Matrix

A Bicomplex matrix $A \in \mathbb{C}_2^{n\times n}$ is said to be i_2 -skew Hermitian matrix if

$(A^t)^\# = (A^\#)^t = -A$

(iii) $i_1 i_2$-Skew-Hermitian Matrix

A Bicomplex matrix $A \in \mathbb{C}_2^{n\times n}$ is said to be $i_1 i_2$-skew Hermitian matrix if

$(A^t)' = (A')^t = -A.$

Note 2.2. Some observations about the definitions are given as follows:

(a) If A is i_1-skew-Hermitian matrix then $A^* = -A^t$.

(b) If A is i_2-skew-Hermitian matrix then $A^\# = -A^t$.

(c) If A is $i_1 i_2$-skew-Hermitian matrix then $A' = -A^t$.

Definition 2.10 (Bicomplex Orthogonal Matrix). A bicomplex matrix $A \in \mathbb{C}_2^{n\times n}$ is said to be orthogonal matrix if $A^tA = AA^t = I$.

Definition 2.11 (Bicomplex Unitary Matrix). There are three types of Bicomplex Unitary Matrix, defined as follows:

(i) i_1-Unitary Matrix

A Bicomplex matrix $A \in \mathbb{C}_2^{n\times n}$ is said to be i_1 -Unitary Matrix if

$A^{*t}A = A^{*t}A = I.$

(ii) i_2-Unitary Matrix

A Bicomplex matrix $A \in \mathbb{C}_2^{n\times n}$ is said to be i_2 -Unitary Matrix if

$A^{\#t}A = A^{\#t}A = I.$

(iii) $i_1 i_2$-Unitary Matrix

A Bicomplex matrix $A \in \mathbb{C}_2^{n\times n}$ is said to be $i_1 i_2$ -Unitary Matrix if

$A'^{t}A = A'^{t}A = I.$

3. Eigenvalues and Eigenvectors of a Bicomplex Matrix

3.1. Eigenvalues of a Bicomplex Matrix. Let $\mathbf{A} = [\xi_{ij}]_{n\times n} \in \mathbb{C}_2^{n\times n}$ and $\boldsymbol{\lambda} \in \mathbb{C}_2$, then the matrix $\mathbf{A} - \boldsymbol{\lambda}\mathbf{I}$ is called the characteristic matrix of $\mathbf{A}$ and $\mathbf{P}(\boldsymbol{\lambda}) = \det(\mathbf{A} - \boldsymbol{\lambda}\mathbf{I})$ is called characteristic polynomial of A.

The equation $\det(A - \lambda I) = 0$ is called the characteristic equation of A and the roots of this equation are called characteristic roots or latent roots or characteristic values or eigenvalues of the matrix A. The set of all eigenvalues of the matrix A is denoted by $A(\lambda)$.

Note 3.1. $A(\lambda) = {}^1A({}^1\lambda)e_1 + {}^2A({}^2\lambda)e_2$ is the collection of all eigenvalues of A iff ${}^1A({}^1\lambda)$and ${}^2A({}^2\lambda)$ are spectrum of ${}^1A$ and ${}^2A$ respectively.

3.2. Eigenvectors of a Bicomplex Matrix. Let $\mathbf{A} = [\xi_{ij}]_{n\times n} \in \mathbb{C}_2^{n\times n}$ and $\boldsymbol{\lambda} \in \mathbb{C}_2$ is a eigenvalue of A, then there exist $\mathbf{X} = {}^1\mathbf{X}\mathbf{e}_1 + {}^2\mathbf{X}\mathbf{e}_2 \in \mathbb{C}_2^{n\times 1}$, ${}^1\mathbf{X} \neq \mathbf{O}$ and ${}^2\mathbf{X} \neq \mathbf{O}$ such that $\mathbf{AX} = \boldsymbol{\lambda}\mathbf{X}$, then X is called characteristic vector or eigenvector of A corresponding to the eigenvalue λ.

3.3. Properties of Eigenvalues of Some Special Matrices.

Proposition 3.1. Let $A \in \mathbb{C}_2^{n\times n}$ and λ is an eigenvalue of A .Then $\alpha\lambda, \alpha \in \mathbb{C}_2$ is an eigenvalue of αA.

Proof: Let X be the eigenvector of the matrix A associated with the eigenvalue λ

$\Rightarrow AX = \lambda X$

Now,

$$
\begin{aligned}
(\alpha A)X &= \alpha(AX) \\
&= \alpha(\lambda X) \\
&= (\alpha\lambda)X
\end{aligned}
$$

$\Rightarrow$ X is an eigenvector of αA for the eigenvalue $\alpha\lambda$.

Proposition 3.2. Let $A \in \mathbb{C}_2^{n\times n}$ and λ is an eigenvalue of A, and $s \geq 0$ is an integer. Then λ^s is an eigenvalue of the matrix A^s.

Proof: Let X be the eigenvector of the matrix A associated with the eigenvalue λ

First, for $s = 0$

$$\begin{aligned} A^sX &= A^0X \\ &= IX \\ &= X \\ &= 1X \\ &= \lambda^0X \end{aligned}$$

The theorem is true for $s = 0$

Now, we assume the theorem is true for s

i.e. $A^sX = \lambda^sX$

Now,

$$\begin{aligned} A^{s+1}X &= A(A^sX) \\ &= A(\lambda^sX) \\ &= \lambda^s(AX) \\ &= \lambda^s(\lambda X) \qquad (\because AX = \lambda X) \\ &= \lambda^{s+1}X \end{aligned}$$

Hence the theorem is true for all $s \geq 0$.

Proposition 3.3. Let $A \in \mathbb{C}_2^{n\times n}$ and λ is an eigenvalue of A. Let $P(\xi)$ be a Bicomplex polynomial in the variable ξ. Then $P(\lambda)$ is an eigenvalue of the matrix $P(A)$.

Proof: Let X be the eigenvector of the matrix A associated with the eigenvalue λ

$\Rightarrow AX = \lambda X$

Also let $P(\xi) = a_n\xi^n + a_{n-1}\xi^{n-1} + \cdots \ldots \ldots . + a_1\xi + a_0$, where $a_j \in \mathbb{C}_2, j = 0,1, \ldots \ldots n$ and ξ is a bicomplex variable

Now,

$$\begin{aligned} P(A)X &= (a_nA^n + a_{n-1}A^{n-1} + \cdots \ldots \ldots . + a_1A + a_0A^0)X \\ &= (a_nA^n)X + (a_{n-1}A^{n-1})X + \cdots \ldots \ldots + (a_0A^0)X \\ &= a_n(A^nX) + a_{n-1}(A^{n-1}X) + \cdots \ldots \ldots + a_0(A^0X) \\ &= a_n(\lambda^nX) + a_{n-1}(\lambda^{n-1}X) + \cdots \ldots \ldots + a_0(\lambda^0X) \end{aligned}$$

$$= \left(a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots \ldots \ldots + a_0\lambda^0\right)X$$

$$= P(\lambda)X$$

Hence $P(\lambda)$ is an eigenvalue of the matrix $P(A)$.

Proposition 3.4. Let $A \in \mathbb{C}_2^{n\times n}$ is a non-singular matrix and λ is an eigenvalue of A . Then λ^{-1} is an eigenvalue of the matrix A^{-1}.

Proof: Let X be the eigenvector of the matrix A associated with the eigenvalue λ

$\Rightarrow AX = \lambda X$

Since the matrix A is non-singular, A^{-1} is exist

$\Rightarrow A^{-1}(AX) = A^{-1}(\lambda X)$

$\Rightarrow (A^{-1}A)X = \lambda(A^{-1}X)$

$\Rightarrow (I)X = \lambda(A^{-1}X)$

$\Rightarrow IX = \lambda(A^{-1}X)$

$\Rightarrow X = \lambda(A^{-1}X)$

$\Rightarrow \lambda(A^{-1}X) = X$

As A is non-singular, λ is also non-singular, $\lambda^{-1} = \frac{1}{\lambda}$ is exist

$\Rightarrow \lambda^{-1}\lambda(A^{-1}X) = \lambda^{-1}X$

$\Rightarrow A^{-1}X = \lambda^{-1}X$

So X is an eigenvector of A^{-1} for the eigenvalue λ^{-1}.

Proposition 3.5. A Bicomplex square matrix A and its transpose A^t have the same eigenvalues.

Proof: Let $A \in \mathbb{C}_2^{n\times n}$, then

$$\det(A - \lambda I) = \det((A - \lambda I)^t)$$
$$= \det(A^t - (\lambda I)^t)$$
$$= \det(A^t - \lambda\, I^t)$$
$$= \det(A^t - \lambda\, I)$$

So, A and A^t have same characteristic polynomial.

Hence A and A^t have the same eigenvalues.

Proposition 3.6. If $\lambda = {}^1\lambda\, e_1 + {}^2\lambda\, e_2$ is the eigenvalue of $A = {}^1A\, e_1 + {}^2A\, e_2 \in \mathbb{H}^{n\times n}$, then $\lambda' = \overline{{}^1\lambda}\, e_1 + \overline{{}^2\lambda}\, e_2$ is also eigenvalue of A.

Proof: Let $\lambda = {}^1\lambda\, e_1 + {}^2\lambda\, e_2$ is the eigenvalue of $A = {}^1A\, e_1 + {}^2A\, e_2 \in \mathbb{H}^{n\times n}$

$\Rightarrow P(\lambda) = 0$

$\Rightarrow \det(A - \lambda I) = 0$

$\Rightarrow \det({}^1A - {}^1\lambda I) = 0$ and $\det({}^2A - {}^2\lambda I) = 0$

$\Rightarrow {}^1P({}^1\lambda) = 0$ and ${}^2P({}^2\lambda) = 0$

$\Rightarrow \overline{{}^1P({}^1\lambda)} = 0$ and $\overline{{}^2P({}^2\lambda)} = 0$

As, ${}^1A\,,{}^2A\ \in \mathbb{C}_0^{n\times n}$

$\Rightarrow \overline{{}^1P({}^1\lambda)} = {}^1P\left(\overline{{}^1\lambda}\right)$ and $\overline{{}^2P({}^2\lambda)} = {}^2P\left(\overline{{}^2\lambda}\right)$

$\Rightarrow {}^1P\left(\overline{{}^1\lambda}\right) = 0$ and ${}^2P\left(\overline{{}^2\lambda}\right) = 0$

$\Rightarrow {}^1P\left(\overline{{}^1\lambda}\right) e_1 + {}^2P\left(\overline{{}^2\lambda}\right) e_2 = 0$

$\Rightarrow P\left(\overline{{}^1\lambda}\, e_1 + \overline{{}^2\lambda}\, e_2\right) = 0$

$\Rightarrow P(\lambda') = 0$

Hence $\lambda' = \overline{{}^1\lambda}\, e_1 + \overline{{}^2\lambda}\, e_2$ is also eigenvalue of A.

Proposition 3.7. Let $X = {}^1X\, e_1 + {}^2X\, e_2$ be the eigenvector of the matrix $A = {}^1A\, e_1 + {}^2A\, e_2 \in \mathbb{H}^{n\times n}$. Then $X' = \overline{{}^1X}\, e_1 + \overline{{}^2X}\, e_2$ is the eigenvector of A for the eigenvalue $\lambda' = \overline{{}^1\lambda}\, e_1 + \overline{{}^2\lambda}\, e_2$.

Proof: Let $X = {}^1X\, e_1 + {}^2X\, e_2$ be the eigenvector of the matrix $A = {}^1A\, e_1 + {}^2A\, e_2 \in \mathbb{H}^{n\times n}$ associated with the eigenvalue λ

$\Rightarrow AX = \lambda X$

$\Rightarrow (AX)' = (\lambda X)'$

$\Rightarrow A'X' = \lambda' X'$

$\Rightarrow AX' = \lambda' X'$ $\quad (\because A = {}^1A\, e_1 + {}^2A\, e_2 \in \mathbb{H}^{n\times n} \Rightarrow A' = A)$

$\Rightarrow X' = \overline{{}^1X}\, e_1 + \overline{{}^2X}\, e_2$ is the eigenvector of A for the eigenvalue $\lambda' = \overline{{}^1\lambda}\, e_1 + \overline{{}^2\lambda}\, e_2$.

Proposition 3.8. Let $A \in \mathbb{H}^{n\times n}$ be a Symmetric matrix and let λ be an eigenvalue of A, then $\lambda' = \lambda$ or equivalently $\lambda \in \mathbb{H}$.

Proof: Let $A = {}^1A\, e_1 + {}^2A\, e_2 \in \mathbb{H}^{n\times n}$ be a symmetric matrix i.e. $A^t = A$

Let $X = {}^1X\, e_1 + {}^2X\, e_2$ be the eigenvector of the matrix $A = {}^1A\, e_1 + {}^2A\, e_2 \in \mathbb{H}^{n\times n}$ associated with the eigenvalue λ

$\Rightarrow AX = \lambda X$

$\Rightarrow (AX)' = (\lambda X)'$

$\Rightarrow A'X' = \lambda' X'$

$\Rightarrow AX' = \lambda' X'$ $\quad (\because A = {}^1A\, e_1 + {}^2A\, e_2 \in \mathbb{H}^{n\times n} \Rightarrow A' = A)$

Again now,

$(AX)^t = (\lambda X)^t$

$\Rightarrow X^t A^t = \lambda\, X^t$

$\Rightarrow X^t A = \lambda\, X^t$ $\quad (\because A^t = A)$

$\Rightarrow (X^t A)X' = (\lambda\, X^t)X'$

$\Rightarrow X^t(AX') = \lambda(X^t X')$

$\Rightarrow X^t(\lambda' X') = \lambda(X^t X')$ $\quad (\because AX' = \lambda' X')$

$\Rightarrow \lambda'(X^t X') = \lambda(X^t X')$

$\Rightarrow (\lambda' - \lambda)X^t X' = 0$

$\Rightarrow \left(\overline{{}^1\lambda} - {}^1\lambda\right) ({}^1X)^t\, \overline{{}^1X} e_1 + \left(\overline{{}^2\lambda} - {}^2\lambda\right) ({}^2X)^t\, \overline{{}^2X} e_2 = 0$

$\Rightarrow \left(\overline{{}^1\lambda} - {}^1\lambda\right) ({}^1X)^t\, \overline{{}^1X} = 0$ and $\left(\overline{{}^2\lambda} - {}^2\lambda\right) ({}^2X)^t\, \overline{{}^2X} = 0$

$\Rightarrow \overline{{}^1\lambda} - {}^1\lambda = 0$ and $\overline{{}^2\lambda} - {}^2\lambda = 0$ $\quad (\because ({}^1X)^t\, \overline{{}^1X} \neq 0$ and $({}^2X)^t\, \overline{{}^2X} \neq 0)$

$\Rightarrow \overline{{}^1\lambda} = {}^1\lambda$ and $\overline{{}^2\lambda} = {}^2\lambda$

$\Rightarrow \lambda = {}^1\lambda e_1 + {}^2\lambda\, e_2 = \overline{{}^1\lambda} e_1 + \overline{{}^2\lambda}\, e_2 = \lambda'$

$\Rightarrow \lambda \in \mathbb{H}$

Proposition 3.9. Let $A \in \mathbb{C}_2^{n\times n}$ be a Bicomplex idempotent matrix and let λ be an eigenvalue of A, then $\lambda \in \{0,1,e_1,e_2\}$.

Proof: Let A be any Bicomplex idempotent matrix i.e. $A^2 = A$

Let $X = {}^1X\, e_1 + {}^2X\, e_2$ be the eigenvector of the matrix A associated with the eigenvalue λ

$\Rightarrow AX = \lambda X$

$\Rightarrow A(AX) = A(\lambda X)$

$\Rightarrow (AA)X = \lambda(AX)$

$\Rightarrow A^2 X = \lambda(AX)$

$\Rightarrow AX = \lambda(AX)$ $\quad (\because A^2 = A)$

$\Rightarrow \lambda X = \lambda(\lambda X)$ $\quad (\because AX = \lambda X)$

$\Rightarrow \lambda X = \lambda^2 X$

$\Rightarrow (\lambda - \lambda^2)X = O$

$\Rightarrow ({}^1\lambda - {}^1\lambda^2)\, {}^1X e_1 + ({}^2\lambda - {}^2\lambda^2)\, {}^2X e_2 = O$

$\Rightarrow ({}^1\lambda - {}^1\lambda^2)\, {}^1X = O$ and $({}^2\lambda - {}^2\lambda^2)\, {}^2X = O$

$\Rightarrow {}^1\lambda - {}^1\lambda^2 = 0$ and ${}^2\lambda - {}^2\lambda^2 = 0$ $\quad (\because {}^1X \neq O$ and ${}^2X \neq O)$

$\Rightarrow {}^1\lambda = 0,1$ and ${}^2\lambda = 0,1$

$\Rightarrow \lambda \in \{0,1,e_1,e_2\}$

Proposition 3.10. Let $A \in \mathbb{C}_2^{n\times n}$ be a Bicomplex skew-idempotent matrix and let λ be an eigenvalue of A, then $\lambda \in \{0,-1,-e_1,-e_2\}$.

Proof: Let A be any Bicomplex Skew-idempotent matrix i.e. $A^2 = -A$

Let $X = {}^1X\, e_1 + {}^2X\, e_2$ be the eigenvector of the matrix A associated with the eigenvalue λ

$\Rightarrow AX = \lambda X$

$\Rightarrow A(AX) = A(\lambda X)$

$\Rightarrow (AA)X = \lambda(AX)$

$\Rightarrow A^2X = \lambda(AX)$

$\Rightarrow -AX = \lambda(AX) \qquad (\because A^2 = -A)$

$\Rightarrow -\lambda X = \lambda(\lambda X) \qquad (\because AX = \lambda X)$

$\Rightarrow -\lambda X = \lambda^2 X$

$\Rightarrow (-\lambda - \lambda^2)X = O$

$\Rightarrow (\lambda + \lambda^2)X = O$

$\Rightarrow ({}^1\lambda + {}^1\lambda^2)\ {}^1Xe_1 + ({}^2\lambda + {}^2\lambda^2)\ {}^2Xe_2 = O$

$\Rightarrow ({}^1\lambda + {}^1\lambda^2)\ {}^1X = O$ and $({}^2\lambda + {}^2\lambda^2)\ {}^2X = O$

$\Rightarrow {}^1\lambda + {}^1\lambda^2 = 0$ and ${}^2\lambda + {}^2\lambda^2 = 0 \qquad (\because {}^1X \neq O$ and ${}^2X \neq O)$

$\Rightarrow {}^1\lambda = 0,-1$ and ${}^2\lambda = 0,-1$

$\Rightarrow \lambda \in \{0,-1,-e_1,-e_2\}$

Proposition 3.11. Let $A \in \mathbb{C}_2^{n\times n}$ such that $A^2 = \eta A, \eta \in \mathbb{C}_2$ and let λ be an eigenvalue of A, then $\lambda \in \{0, \eta, {}^1\eta e_1, {}^2\eta e_2\}$.

Proof: Let $A \in \mathbb{C}_2^{n\times n}$ such that $A^2 = \eta A, \eta \in \mathbb{C}_2$

Let $X = {}^1X\, e_1 + {}^2X\, e_2$ be the eigenvector of the matrix A associated with the eigenvalue λ

$\Rightarrow AX = \lambda X$

$\Rightarrow A(AX) = A(\lambda X)$

$\Rightarrow (AA)X = \lambda(AX)$

$\Rightarrow A^2X = \lambda(AX)$

$\Rightarrow (\eta A)X = \lambda(AX) \qquad (\because A^2 = \eta A)$

$\Rightarrow \eta(AX) = \lambda(AX)$

$\Rightarrow \eta\lambda X = \lambda(\lambda X) \qquad (\because AX = \lambda X)$

$\Rightarrow \eta\lambda X = \lambda^2 X$

$\Rightarrow (\eta\lambda - \lambda^2)X = O$

$\Rightarrow ({}^1\eta{}^1\lambda - {}^1\lambda^2)\ {}^1Xe_1 + ({}^2\eta{}^2\lambda - {}^2\lambda^2)\ {}^2Xe_2 = O$

$\Rightarrow ({}^1\eta{}^1\lambda - {}^1\lambda^2){}^1X = O$ and $({}^2\eta{}^2\lambda - {}^2\lambda^2){}^2X = O$

$\Rightarrow {}^1\eta{}^1\lambda - {}^1\lambda^2 = 0$ and ${}^2\eta{}^2\lambda - {}^2\lambda^2 = 0$ $\quad (\because {}^1X \neq O$ and ${}^2X \neq O)$

$\Rightarrow {}^1\lambda = 0, {}^1\eta$ and ${}^2\lambda = 0, {}^2\eta$

$\Rightarrow \lambda \in \{0, \eta, {}^1\eta e_1, {}^2\eta e_2\}$

Note 3.2.

(i) For $\eta = 1$, we get **Proposition 3.9**

(ii) For $\eta = -1$, we get **Proposition 3.10**

Proposition 3.12. Let $A \in \mathbb{C}_2^{n\times n}$ be a Bicomplex involutory matrix and let λ be an eigenvalue of A, then $\lambda \in \{1, -1, i_1 i_2, -i_1 i_2\}$.

Proof: Let A be any Bicomplex involutory matrix i.e. $A^2 = I$

Let $X = {}^1X\, e_1 + {}^2X\, e_2$ be the eigenvector of the matrix A associated with the eigenvalue λ

$\Rightarrow AX = \lambda X$

$\Rightarrow A(AX) = A(\lambda X)$

$\Rightarrow (AA)X = \lambda(AX)$

$\Rightarrow A^2X = \lambda(AX)$

$\Rightarrow IX = \lambda(AX) \quad (\because A^2 = I)$

$\Rightarrow X = \lambda(\lambda X) \quad (\because AX = \lambda X)$

$\Rightarrow X = \lambda^2 X$

$\Rightarrow (1 - \lambda^2)X = O$

$\Rightarrow (1 - {}^1\lambda^2)\ {}^1Xe_1 + (1 - {}^2\lambda^2)\ {}^2Xe_2 = O$

$\Rightarrow (1 - {}^1\lambda^2)\ {}^1X = O$ and $(1 - {}^2\lambda^2)\ {}^2X = O$

$\Rightarrow 1 - {}^1\lambda^2 = 0$ and $1 - {}^2\lambda^2 = 0$ $\quad (\because {}^1X \neq O$ and ${}^2X \neq O)$

$\Rightarrow {}^1\lambda = 1, -1$ and ${}^2\lambda = 1, -1$

$\Rightarrow \lambda \in \{1, -1, i_1 i_2, -i_1 i_2\}$

Proposition 3.13. Let $A \in \mathbb{C}_2^{n\times n}$ be a Bicomplex Skew-involutory matrix and let λ be an eigenvalue of A, then $\lambda \in \{i_1, -i_1, i_2, -i_2\}$.

Proof: Let $A \in \mathbb{C}_2^{n\times n}$ be a Bicomplex Skew-involutory matrix i.e. $A^2 = -I$

Let X be the eigenvector of the matrix A associated with the eigenvalue λ

$\Rightarrow AX = \lambda X$

$\Rightarrow A(AX) = A(\lambda X)$

$\Rightarrow (AA)X = \lambda(AX)$

$\Rightarrow A^2X = \lambda(AX)$

$\Rightarrow (-I)X = \lambda(AX) \qquad (\because A^2 = -I)$

$\Rightarrow -(IX) = \lambda(AX)$

$\Rightarrow -X = \lambda(AX)$

$\Rightarrow -X = \lambda(\lambda X) \qquad (\because AX = \lambda X)$

$\Rightarrow -X = \lambda^2 X$

$\Rightarrow (-1 - \lambda^2)X = O$

$\Rightarrow (1 + \lambda^2)X = O$

$\Rightarrow (1 + {}^1\lambda^2)\, {}^1Xe_1 + (1 + {}^2\lambda^2)\, {}^2Xe_2 = O$

$\Rightarrow (1 + {}^1\lambda^2)\, {}^1X = O$ and $(1 + {}^2\lambda^2)\, {}^2X = O$

$\Rightarrow 1 + {}^1\lambda^2 = 0$ and $1 + {}^2\lambda^2 = 0 \qquad (\because {}^1X \neq O$ and ${}^2X \neq O)$

$\Rightarrow {}^1\lambda = i_1, -i_1$ and ${}^2\lambda = i_1, -i_1$

$\Rightarrow \lambda \in \{i_1, -i_1, i_2, -i_2\}$

Proposition 3.14. Let $A \in \mathbb{C}_2^{n\times n}$ such that $A^2 = \eta I, \eta \in C_2$ and let λ be an eigenvalue of A, then

$\lambda \in \left\{ \sqrt{{}^1\eta}\, e_1 + \sqrt{{}^2\eta}\, e_2, -\left(\sqrt{{}^1\eta}\, e_1 + \sqrt{{}^2\eta}\, e_2\right), \sqrt{{}^1\eta}\, e_1 - \sqrt{{}^2\eta}\, e_2, -\sqrt{{}^1\eta}\, e_1 + \sqrt{{}^2\eta}\, e_2 \right\}.$

Proof: Let $A \in \mathbb{C}_2^{n\times n}$ be any Bicomplex matrix such that $A^2 = \eta I$

Let X be the eigenvector of the matrix A associated with the eigenvalue λ

$\Rightarrow AX = \lambda X$

$\Rightarrow A(AX) = A(\lambda X)$

$\Rightarrow (AA)X = \lambda(AX)$

$\Rightarrow A^2X = \lambda(AX)$

$\Rightarrow (\eta I)X = \lambda(AX) \qquad (\because A^2 = \eta I)$

$\Rightarrow \eta(IX) = \lambda(AX)$

$\Rightarrow \eta X = \lambda(AX)$

$\Rightarrow \eta X = \lambda(\lambda X) \qquad (\because AX = \lambda X)$

$\Rightarrow \eta X = \lambda^2 X$

$\Rightarrow (\eta - \lambda^2)X = O$

$\Rightarrow ({}^1\eta - {}^1\lambda^2)\ {}^1Xe_1 + ({}^2\eta - {}^2\lambda^2)\ {}^2Xe_2 = O$

$\Rightarrow ({}^1\eta - {}^1\lambda^2)\ {}^1X = O$ and $({}^2\eta - {}^2\lambda^2)\ {}^2X = O$

$\Rightarrow {}^1\eta - {}^1\lambda^2 = 0$ and ${}^2\eta - {}^2\lambda^2 = 0 \qquad (\because {}^1X \neq O$ and ${}^2X \neq O)$

$\Rightarrow {}^1\lambda = \sqrt{{}^1\eta}, -\sqrt{{}^1\eta}$ and ${}^2\lambda = \sqrt{{}^2\eta}, -\sqrt{{}^2\eta}$

$\Rightarrow \lambda \in \left\{\sqrt{{}^1\eta}\, e_1 + \sqrt{{}^2\eta}\, e_2, -\left(\sqrt{{}^1\eta}\, e_1 + \sqrt{{}^2\eta}\, e_2\right), \sqrt{{}^1\eta}\, e_1 - \sqrt{{}^2\eta}\, e_2, -\sqrt{{}^1\eta}\, e_1 + \sqrt{{}^2\eta}\, e_2\right\}$

Note 3.3.

(i) For $\eta = 1$, we get **Proposition 3.12**

(ii) For $\eta = -1$, we get **Proposition 3.13**

Proposition 3.15:

(i) Let $A \in \mathbb{C}_2^{n\times n}$ be a $i_1 -$ Hermitian Matrix and λ is a eigenvalue of A with corresponding eigenvector $X = {}^1X\, e_1 + {}^2X\, e_2$, such that $\left(\overline{{}^2X}\right)^t {}^1X \neq 0$. Then $\lambda = \lambda^*$or equivalently $\lambda \in \mathbb{C}(i_2)$.

(ii) Let $A \in \mathbb{C}_2^{n\times n}$ be a $i_1 -$ Skew $-$ Hermitian Matrix and λ is a eigenvalue of A with corresponding eigenvector $X = {}^1X\, e_1 + {}^2X\, e_2$, such that $\left(\overline{{}^2X}\right)^t {}^1X \neq 0$. Then $\lambda = -\lambda^*$or equivalently $\lambda \in i_1\mathbb{C}(i_2)$.

Proof: (i) Let $A \in \mathbb{C}_2^{n\times n}$ be a $i_1 -$ Hermitian Matrix i.e. $A^{*t} = A$

Let $X = {}^1X\, e_1 + {}^2X\, e_2$ be the eigenvector of the matrix A associated with the eigenvalue λ such that $\left(\overline{{}^2X}\right)^t {}^1X \neq 0$

$\Rightarrow AX = \lambda X$

$\Rightarrow (AX)^{*t} = (\lambda X)^{*t}$

$\Rightarrow X^{*t}A^{*t} = \lambda^*\, X^{*t}$

$\Rightarrow X^{*t}A = \lambda^*\, X^{*t} \qquad$ (As A is $i_1 -$ Hermitian matrix $A^{*t} = A$)

$\Rightarrow (X^{*t}A)X = (\lambda^*\, X^{*t})X$

$\Rightarrow X^{*t}(AX) = \lambda^*(X^{*t}X)$

$\Rightarrow X^{*t}(\lambda X) = \lambda^*(X^{*t}X) \qquad (\because AX = \lambda X)$

$\Rightarrow \lambda(X^{*t}X) = \lambda^*(X^{*t}X)$

$\Rightarrow (\lambda - \lambda^*)X^{*t}X = 0$

$\Rightarrow \left({}^1\lambda - \overline{{}^2\lambda}\right)\left(\overline{{}^2X}\right)^t {}^1X\, e_1 + \left({}^2\lambda - \overline{{}^1\lambda}\right)\left(\overline{{}^1X}\right)^t {}^2X\, e_2 = 0$

$\Rightarrow \left({}^1\lambda - \overline{{}^2\lambda}\right)\left(\overline{{}^2X}\right)^t {}^1X = O$ and $\left({}^2\lambda - \overline{{}^1\lambda}\right)\left(\overline{{}^1X}\right)^t {}^2X = 0$

$\Rightarrow {}^1\lambda - \overline{{}^2\lambda} = 0$ and ${}^2\lambda - \overline{{}^1\lambda} = 0 \qquad \left(\because \left(\overline{{}^2X}\right)^t {}^1X \neq 0 \Rightarrow \left(\overline{{}^1X}\right)^t {}^2X \neq 0\right)$

$\Rightarrow {}^1\lambda = \overline{{}^2\lambda}$ and ${}^2\lambda = \overline{{}^1\lambda}$

$\Rightarrow \lambda = {}^1\lambda\, e_1 + {}^2\lambda\, e_2 = \overline{{}^2\lambda}\, e_1 + \overline{{}^1\lambda}\, e_2 = \lambda^*$

$\Rightarrow \lambda \in \mathbb{C}(i_2)$

(ii) Let $A \in \mathbb{C}_2^{n\times n}$ be a $i_1 -$ Skew $-$ Hermitian Matrix i.e. $A^{*t} = -A$

Let $X = {}^1X\, e_1 + {}^2X\, e_2$ be the eigenvector of the matrix A associated with the eigenvalue λ such that $\left(\overline{{}^2X}\right)^t {}^1X \neq 0$

$\Rightarrow AX = \lambda X$

$\Rightarrow (AX)^{*t} = (\lambda X)^{*t}$

$\Rightarrow X^{*t}A^{*t} = \lambda^* X^{*t}$

$\Rightarrow X^{*t}(-A) = \lambda^* X^{*t}$ (As A is $i_1 -$ Skew $-$ Hermitian matrix $A^{*t} = -A$)

$\Rightarrow -(X^{*t}A)X = (\lambda^* X^{*t})X$

$\Rightarrow -X^{*t}(AX) = \lambda^*(X^{*t} X)$

$\Rightarrow -X^{*t}(\lambda X) = \lambda^*(X^{*t} X)$ ($\because AX = \lambda X$)

$\Rightarrow -\lambda(X^{*t}X) = \lambda^*(X^{*t} X)$

$\Rightarrow (\lambda + \lambda^*)X^{*t}X = 0$

$\Rightarrow \left({}^1\lambda + \overline{{}^2\lambda}\right)\left(\overline{{}^2X}\right)^t {}^1X\, e_1 + \left({}^2\lambda + \overline{{}^1\lambda}\right)\left(\overline{{}^1X}\right)^t {}^2X\, e_2 = 0$

$\Rightarrow \left({}^1\lambda + \overline{{}^2\lambda}\right)\left(\overline{{}^2X}\right)^t {}^1X = 0$ and $\left({}^2\lambda + \overline{{}^1\lambda}\right)\left(\overline{{}^1X}\right)^t {}^2X = 0$

$\Rightarrow {}^1\lambda + \overline{{}^2\lambda} = 0$ and ${}^2\lambda + \overline{{}^1\lambda} = 0$ $\left(\because \left(\overline{{}^2X}\right)^t {}^1X \neq 0 \Rightarrow \left(\overline{{}^1X}\right)^t {}^2X \neq 0\right)$

$\Rightarrow {}^1\lambda = -\overline{{}^2\lambda}$ and ${}^2\lambda = -\overline{{}^1\lambda}$

$\Rightarrow \lambda = {}^1\lambda\, e_1 + {}^2\lambda\, e_2 = -\left(\overline{{}^2\lambda}\, e_1 + \overline{{}^1\lambda}\, e_2\right) = -\lambda^*$

$\Rightarrow \lambda \in i_1\mathbb{C}(i_2)$

Proposition 3.16:

(i) Let $A \in \mathbb{C}_2^{n\times n}$ be a $i_2 -$ Hermitian Matrix and λ is a eigenvalue of A with corresponding eigenvector $X = {}^1X\, e_1 + {}^2X\, e_2$, such that $({}^2X)^t\, {}^1X \neq 0$. Then $\lambda = \lambda^{\#}$ or equivalently $\lambda \in \mathbb{C}(i_1)$.

(ii) Let $A \in \mathbb{C}_2^{n\times n}$ be a $i_2 -$ Skew $-$ Hermitian Matrix and λ is a eigenvalue of A with corresponding eigenvector $X = {}^1X\, e_1 + {}^2X\, e_2$, such that $({}^2X)^t\, {}^1X \neq 0$. Then $\lambda = -\lambda^{\#}$ or equivalently $\lambda \in i_2\mathbb{C}(i_1)$.

Proof: (i) Let $A \in \mathbb{C}_2^{n\times n}$ be a $i_2 -$ Hermitian Matrix i.e. $A^{\#t} = A$

Let $X = {}^1X\, e_1 + {}^2X\, e_2$ be the eigenvector of the matrix A associated with the eigenvalue λ such that $({}^2X)^t\, {}^1X \neq 0$

$\Rightarrow AX = \lambda X$

$\Rightarrow (AX)^{\#t} = (\lambda X)^{\#t}$

$\Rightarrow X^{\#t}A^{\#t} = \lambda^{\#}\, X^{\#t}$

$\Rightarrow X^{\#t}A = \lambda^{\#}\, X^{\#t}$ (As A is $i_2 -$ Hermitian matrix $A^{\#t} = A$)

$\Rightarrow (X^{\#t}A)X = (\lambda^{\#}\, X^{\#t})X$

$\Rightarrow X^{\#t}(AX) = \lambda^{\#}(X^{\#t}X)$

$\Rightarrow X^{\#t}(\lambda X) = \lambda^{\#}(X^{\#t}X)$ ($\because AX = \lambda X$)

$\Rightarrow \lambda(X^{\#t}X) = \lambda^{\#}(X^{\#t}X)$

$\Rightarrow (\lambda - \lambda^{\#})X^{\#t}X = 0$

$\Rightarrow ({}^1\lambda - {}^2\lambda)\,({}^2X)^{t}\,{}^1X\, e_1 + ({}^2\lambda - {}^1\lambda)({}^1X)^{t}\,{}^2X\, e_2 = 0$

$\Rightarrow ({}^1\lambda - {}^2\lambda)\,({}^2X)^{t}\,{}^1X = 0$ and $({}^2\lambda - {}^1\lambda)({}^1X)^{t}\,{}^2X = 0$

$\Rightarrow {}^1\lambda - {}^2\lambda = 0$ and ${}^2\lambda - {}^1\lambda = 0$ $(({}^2X)^{t}\,{}^1X = ({}^1X)^{t}\,{}^2X \neq 0)$

$\Rightarrow {}^1\lambda = {}^2\lambda$

$\Rightarrow \lambda = {}^1\lambda e_1 + {}^2\lambda\, e_2 = {}^2\lambda\, e_1 + {}^1\lambda\, e_2 = \lambda^{\#}$

$\Rightarrow \lambda \in \mathbb{C}(i_1)$

(ii) Let $A \in \mathbb{C}_2^{n\times n}$ be a $i_2 -$ Skew $-$ Hermitian Matrix i.e. $A^{\#t} = -A$

Let $X = {}^1X\, e_1 + {}^2X\, e_2$ be the eigenvector of the matrix A associated with the eigenvalue λ such that $({}^2X)^{t}\,{}^1X \neq 0$

$\Rightarrow AX = \lambda X$

$\Rightarrow (AX)^{\#t} = (\lambda X)^{\#t}$

$\Rightarrow X^{\#t}A^{\#t} = \lambda^{\#}\, X^{\#t}$

$\Rightarrow X^{\#t}(-A) = \lambda^{\#}\, X^{\#t}$ (As A is $i_2 -$ Skew $-$ Hermitian Matrix $A^{\#t} = -A$)

$\Rightarrow -X^{\#t}A = \lambda^{\#}\, X^{\#t}$

$\Rightarrow -(X^{\#t}A)X = (\lambda^{\#}\, X^{\#t})X$

$\Rightarrow -X^{\#t}(AX) = \lambda^{\#}(X^{\#t}X)$

$\Rightarrow -X^{\#t}(\lambda X) = \lambda^{\#}(X^{\#t}X)$ ($\because AX = \lambda X$)

$\Rightarrow -\lambda(X^{\#t}X) = \lambda^{\#}(X^{\#t}X)$

$\Rightarrow (-\lambda - \lambda^{\#})X^{\#t}X = 0$

$\Rightarrow (\lambda + \lambda^{\#})X^{\#t}X = 0$

$\Rightarrow ({}^1\lambda + {}^2\lambda)\,({}^2X)^{t}\,{}^1X\, e_1 + ({}^2\lambda + {}^1\lambda)({}^1X)^{t}\,{}^2X\, e_2 = 0$

$\Rightarrow ({}^1\lambda + {}^2\lambda)\,({}^2X)^{t}\,{}^1X = 0$ and $({}^2\lambda + {}^1\lambda)({}^1X)^{t}\,{}^2X = 0$

$\Rightarrow {}^1\lambda + {}^2\lambda = 0$ and ${}^2\lambda + {}^1\lambda = 0$ $(({}^2X)^{t}\,{}^1X = ({}^1X)^{t}\,{}^2X \neq 0)$

$\Rightarrow {}^1\lambda = -{}^2\lambda$

$\Rightarrow \lambda = {}^1\lambda e_1 + {}^2\lambda\, e_2 = -{}^2\lambda\, e_1 - {}^1\lambda\, e_2 = -\lambda^{\#}$

$\Rightarrow \lambda \in i_2\mathbb{C}(i_1)$

Proposition 3.17:

(i) Bicomplex $i_1 i_2 -$ Hermitian Matrix have Hyperbolic Eigenvalues.

Suppose $A \in \mathbb{C}_2^{n\times n}$ be a $i_1 i_2 -$ Hermitian Matrix and λ is a eigenvalue of A. Then $\lambda = \lambda'$ or equivalently $\lambda \in \mathbb{H}$.

(ii) Suppose $A \in \mathbb{C}_2^{n\times n}$ be a $i_1 i_2 -$ Skew $-$ Hermitian Matrix and λ is a eigenvalue of A.

Then $\lambda = -\lambda'$ or equivalently $\lambda \in i_1\mathbb{H}$ or $i_2\mathbb{H}$.

Proof: (i) Let $A \in \mathbb{C}_2^{n\times n}$ be a $i_1 i_2 -$ Hermitian Matrix i.e. $A'^t = A$

Let $X = {}^1X\, e_1 + {}^2X\, e_2$ be the eigenvector of the matrix A associated with the eigenvalue λ

$\Rightarrow AX = \lambda X$

$\Rightarrow (AX)'^t = (\lambda X)'^t$

$\Rightarrow X'^t A'^t = \lambda' \, X'^t$

$\Rightarrow X'^t A = \lambda' \, X'^t$ (As A is $i_1 i_2 -$ Hermitian matrix $A'^t = A$)

$\Rightarrow (X'^t A)X = (\lambda' \, X'^t)X$

$\Rightarrow X'^t (AX) = \lambda' (X'^t X)$

$\Rightarrow X'^t (\lambda X) = \lambda' (X'^t X)$ ($\because AX = \lambda X$)

$\Rightarrow \lambda(X'^t X) = \lambda' (X'^t X)$

$\Rightarrow (\lambda - \lambda')X'^t X = 0$

$\Rightarrow \left({}^1\lambda - \overline{{}^1\lambda}\right) \left(\overline{{}^1X} \right)^t {}^1X\, e_1 + \left({}^2\lambda - \overline{{}^2\lambda}\right)\left(\overline{{}^2X} \right)^t {}^2X\, e_2 = 0$

$\Rightarrow \left({}^1\lambda - \overline{{}^1\lambda}\right) \left(\overline{{}^1X} \right)^t {}^1X = 0$ and $\left({}^2\lambda - \overline{{}^2\lambda}\right)\left(\overline{{}^2X} \right)^t {}^2X = 0$

$\Rightarrow {}^1\lambda - \overline{{}^1\lambda} = 0$ and ${}^2\lambda - \overline{{}^2\lambda} = 0$ ($\because \left(\overline{{}^1X} \right)^t {}^1X \neq 0$ and $\left(\overline{{}^2X} \right)^t {}^2X \neq 0$)

$\Rightarrow {}^1\lambda = \overline{{}^1\lambda}$ and ${}^2\lambda = \overline{{}^2\lambda}$

$\Rightarrow \lambda = {}^1\lambda\, e_1 + {}^2\lambda\, e_2 = \overline{{}^1\lambda}\, e_1 + \overline{{}^2\lambda}\, e_2 = \lambda'$

$\Rightarrow \lambda \in \mathbb{H}$

(ii) Let $A \in \mathbb{C}_2^{n\times n}$ be a $i_1 i_2 -$ Skew $-$ Hermitian Matrix

i.e. $A'^t = -A$

Let $X = {}^1X\, e_1 + {}^2X\, e_2$ be the eigenvector of the matrix A associated with the eigenvalue λ

$\Rightarrow AX = \lambda X$

$\Rightarrow (AX)'^t = (\lambda X)'^t$

$\Rightarrow X'^t A'^t = \lambda' \, X'^t$

$\Rightarrow X'^t (-A) = \lambda' \, X'^t$ (As A is $i_1 i_2 -$ Skew $-$ Hermitian matrix $A'^t = -A$)

$\Rightarrow -(X'^{t}A)X = (\lambda'\ X'^{t}\ )X$

$\Rightarrow -X'^{t}(AX) = \lambda'(\ X'^{t}\ X\ )$

$\Rightarrow -X'^{t}(\lambda X) = \lambda'(\ X'^{t}\ X\ )$ $\qquad (\because AX = \lambda X)$

$\Rightarrow -\lambda(X'^{t}X) = \lambda'(\ X'^{t}\ X\ )$

$\Rightarrow (\lambda + \lambda')X'^{t}X = 0$

$\Rightarrow \left({}^{1}\lambda + \overline{{}^{1}\lambda}\right)\left(\overline{{}^{1}X}\right)^{t}{}^{1}X\,e_1 + \left({}^{2}\lambda + \overline{{}^{2}\lambda}\right)\left(\overline{{}^{2}X}\right)^{t}{}^{2}X\,e_2 = 0$

$\Rightarrow \left({}^{1}\lambda + \overline{{}^{1}\lambda}\right)\left(\overline{{}^{1}X}\right)^{t}{}^{1}X = 0$ and $\left({}^{2}\lambda + \overline{{}^{2}\lambda}\right)\left(\overline{{}^{2}X}\right)^{t}{}^{2}X = 0$

$\Rightarrow {}^{1}\lambda + \overline{{}^{1}\lambda} = 0$ and ${}^{2}\lambda + \overline{{}^{2}\lambda} = 0$ $\qquad \left(\because \left(\overline{{}^{1}X}\right)^{t}{}^{1}X \neq 0\right.$ and $\left.\left(\overline{{}^{2}X}\right)^{t}{}^{2}X \neq 0\right)$

$\Rightarrow {}^{1}\lambda = -\overline{{}^{1}\lambda}$ and ${}^{2}\lambda = -\overline{{}^{2}\lambda}$

$\Rightarrow \lambda = {}^{1}\lambda\,e_1 + {}^{2}\lambda\,e_2 = -\left(\overline{{}^{1}\lambda}\,e_1 + \overline{{}^{2}\lambda}\,e_2\right) = -\lambda'$

$\Rightarrow \lambda \in i_1\mathbb{H}$ or $i_2\mathbb{H}$

Proposition 3.18: Let $A \in \mathbb{C}_2^{n\times n}$ be a $i_1 -$ Unitary Matrix and λ is a eigenvalue of A with corresponding eigenvector $X = {}^{1}X\,e_1 + {}^{2}X\,e_2$, such that $\left(\overline{{}^{2}X}\right)^{t}{}^{1}X \neq 0$. Then $\lambda = (\lambda^{-1})^{*}$.

Proof: Let $A \in \mathbb{C}_2^{n\times n}$ be a $i_1 -$ Unitary Matrix i.e. $A^{*t}A = AA^{*t} = I$

Let $X = {}^{1}X\,e_1 + {}^{2}X\,e_2$ be the eigenvector of the matrix A associated with the eigenvalue λ

$\Rightarrow AX = \lambda X$

$\Rightarrow (AX)^{*t} = (\lambda X)^{*t}$

$\Rightarrow X^{*t}A^{*t} = \lambda^{*}\ X^{*t}$

$\Rightarrow (X^{*t}A^{*t})AX = (\lambda^{*}\ X^{*t})\lambda X$ $\qquad (\because AX = \lambda X)$

$\Rightarrow X^{*t}(A^{*t}A)X = \lambda^{*}\lambda(\ X^{*t}X)$

$\Rightarrow X^{*t}(I)X = \lambda^{*}\lambda(\ X^{*t}X)$ $\qquad (\because A^{*t}A = AA^{*t} = I\ )$

$\Rightarrow X^{*t}X = \lambda^{*}\lambda(\ X^{*t}X)$

$\Rightarrow (1 - \lambda^{*}\lambda)X^{*t}X = 0$

$\Rightarrow \left(1 - \overline{{}^{2}\lambda}\,{}^{1}\lambda\right)\left(\overline{{}^{2}X}\right)^{t}{}^{1}X\,e_1 + \left(1 - \overline{{}^{1}\lambda}\,{}^{2}\lambda\right)\left(\overline{{}^{1}X}\right)^{t}{}^{2}X\,e_2 = 0$

$\Rightarrow \left(1 - \overline{{}^{2}\lambda}\,{}^{1}\lambda\right)\left(\overline{{}^{2}X}\right)^{t}{}^{1}X = 0$ and $\left(1 - \overline{{}^{1}\lambda}\,{}^{2}\lambda\right)\left(\overline{{}^{1}X}\right)^{t}{}^{2}X = 0$

$\Rightarrow 1 - \overline{{}^{2}\lambda}\,{}^{1}\lambda = 0$ and $1 - \overline{{}^{1}\lambda}\,{}^{2}\lambda = 0$ $\qquad \left(\left(\overline{{}^{2}X}\right)^{t}{}^{1}X \neq 0 \Rightarrow \left(\overline{{}^{1}X}\right)^{t}{}^{2}X \neq 0\right)$

$\Rightarrow \overline{{}^{2}\lambda}\,{}^{1}\lambda = 1$ and $\overline{{}^{1}\lambda}\,{}^{2}\lambda = 1$

$\Rightarrow {}^{1}\lambda = \frac{1}{\overline{{}^{2}\lambda}}$ and ${}^{2}\lambda = \frac{1}{\overline{{}^{1}\lambda}}$

$$\Rightarrow \lambda = {}^{1}\lambda\,e_1 + {}^{2}\lambda\,e_2 = \frac{1}{\overline{{}^{2}\lambda}}e_1 + \frac{1}{\overline{{}^{1}\lambda}}e_2 = \frac{1}{\lambda^{*}} = (\lambda^{-1})^{*}$$

Note 3.4. λ is a non-singular and λ^* is a multiplicative inverse of λ.

Proposition 3.19: Let $A \in \mathbb{C}_2^{n\times n}$ be a $i_2 -$ Unitary Matrix and λ is a eigenvalue of A with corresponding eigenvector $X = {}^1X\, e_1 + {}^2X\, e_2$, such that $({}^2X)^{t\,1}X \neq 0$. Then $\lambda = (\lambda^{-1})^{\#}$.

Proof: Let $A \in \mathbb{C}_2^{n\times n}$ be a $i_2 -$ Unitary Matrix i.e. $A^{\#t}A = AA^{\#t} = I$

Let $X = {}^1X\, e_1 + {}^2X\, e_2$ be the eigenvector of the matrix A associated with the eigenvalue λ such that $({}^2X)^{t\,1}X \neq 0$

$\Rightarrow AX = \lambda X$

$\Rightarrow (AX)^{\#t} = (\lambda X)^{\#t}$

$\Rightarrow X^{\#t}A^{\#t} = \lambda^{\#}\, X^{\#t}$

$\Rightarrow (X^{\#t}A^{\#t})AX = (\lambda^{\#}\, X^{\#t})\lambda X \qquad (\because AX = \lambda X)$

$\Rightarrow X^{\#t}(A^{\#t}A)X = \lambda^{\#}\lambda(X^{\#t}X)$

$\Rightarrow X^{\#t}(I)X = \lambda^{\#}\lambda(X^{\#t}X) \qquad (\because A^{\#t}A = AA^{\#t} = I)$

$\Rightarrow X^{\#t}X = \lambda^{\#}\lambda(X^{\#t}X)$

$\Rightarrow (1 - \lambda^{\#}\lambda)X^{\#t}X = 0$

$\Rightarrow (1 - {}^2\lambda^1\lambda)\, ({}^2X)^{t\,1}X\, e_1 + (1 - {}^1\lambda^2\lambda)({}^1X)^{t\,2}X\, e_2 = 0$

$\Rightarrow (1 - {}^2\lambda^1\lambda)\, ({}^2X)^{t\,1}X = 0$ and $(1 - {}^1\lambda^2\lambda)({}^1X)^{t\,2}X = 0$

$\Rightarrow 1 - {}^2\lambda^1\lambda = 0$ and $1 - {}^1\lambda^2\lambda = 0 \qquad (({}^2X)^{t\,1}X = ({}^1X)^{t\,2}X \neq 0)$

$\Rightarrow {}^2\lambda^1\lambda = 1$

$\Rightarrow {}^1\lambda = \frac{1}{{}^2\lambda}$ and ${}^2\lambda = \frac{1}{{}^1\lambda}$

$\Rightarrow \lambda = {}^1\lambda\, e_1 + {}^2\lambda\, e_2 \;= \frac{1}{{}^2\lambda} e_1 + \frac{1}{{}^1\lambda} e_2 = \frac{1}{\lambda^{\#}} = (\lambda^{-1})^{\#}$

Note 3.5. λ is non-singular and $\lambda^{\#}$ is a multiplicative inverse of λ.

Proposition 3.20. Let $A \in \mathbb{C}_2^{n\times n}$ be a $i_1 i_2 -$ Unitary Matrix and λ is a eigenvalue of A. Then $\lambda = (\lambda^{-1})'$.

Proof: Let $A \in \mathbb{C}_2^{n\times n}$ be a $i_1 i_2 -$ Unitary Matrix i.e. $A'^{t}A = AA'^{t} = I$

Let $X = {}^1X\, e_1 + {}^2X\, e_2$ be the eigenvector of the matrix A associated with the eigenvalue λ

$\Rightarrow AX = \lambda X$

$\Rightarrow (AX)'^{t} = (\lambda X)'^{t}$

$\Rightarrow X'^{t}A'^{t} = \lambda'\, X'^{t}$

$\Rightarrow (X'^{t}A'^{t})AX = (\lambda'\, X'^{t})\lambda X \qquad (\because AX = \lambda X)$

$\Rightarrow X'^{t}(A'^{t}A)X = \lambda'\lambda(X'^{t}X)$

$\Rightarrow X'^{t}(I)X = \lambda'\lambda(X'^{t}X)$ $\quad (\because A'^{t}A = AA'^{t} = I)$

$\Rightarrow X'^{t}X = \lambda'\lambda(X'^{t}X)$

$\Rightarrow (1-\lambda'\lambda)X'^{t}X = 0$

$\Rightarrow \left(1-\overline{^1\lambda}\,{}^1\lambda\right)\left(\overline{^1X}\right)^{t}{}^1X\,e_1 + \left(1-\overline{^2\lambda}\,{}^2\lambda\right)\left(\overline{^2X}\right)^{t}{}^2X\,e_2 = 0$

$\Rightarrow \left(1-\overline{^1\lambda}\,{}^1\lambda\right)\left(\overline{^1X}\right)^{t}{}^1X = O$ and $\left(1-\overline{^2\lambda}\,{}^2\lambda\right)\left(\overline{^2X}\right)^{t}{}^2X = 0$

$\Rightarrow 1-\overline{^1\lambda}\,{}^1\lambda = 0$ and $1-\overline{^2\lambda}\,{}^2\lambda = 0$ $\quad \left(\because \left(\overline{^1X}\right)^{t}{}^1X \neq 0 \text{ and } \left(\overline{^2X}\right)^{t}{}^2X \neq 0\right)$

$\Rightarrow \overline{^1\lambda}\,{}^1\lambda = 1$ and $\overline{^2\lambda}\,{}^2\lambda = 1$

$\Rightarrow {}^1\lambda = \frac{1}{\overline{^1\lambda}}$ and ${}^2\lambda = \frac{1}{\overline{^2\lambda}}$

$\Rightarrow \lambda = {}^1\lambda\,e_1 + {}^2\lambda\,e_2 = \frac{1}{\overline{^1\lambda}}e_1 + \frac{1}{\overline{^2\lambda}}e_2 = \frac{1}{\lambda'} = (\lambda^{-1})'$

Note 3.6. λ is non-singular and λ' is a multiplicative inverse of λ.

Proposition 3.21: Let $A \in \mathbb{C}_2^{n\times n}$ be a i_1i_2-Hermitian Matrix and X and Y are two eigenvectors of A associated with the eigenvalues λ and μ such that $\lambda-\mu \notin O_2$. Then $X'Y = 0$.

Proof: As A is i_1i_2-Hermitian Matrix, $A' = A$ and $\lambda' = \lambda$, $\mu' = \mu$

Let X and Y are the eigenvectors of the matrix A associated with the eigenvalues λ and μ respectively

$\Rightarrow AX = \lambda X$ and $AY = \mu Y$

Now,

$AX = \lambda X$

$\Rightarrow (AX)' = (\lambda X)'$

$\Rightarrow X'A' = \lambda'X'$

$\Rightarrow X'A = \lambda X'$ $\quad (\because A' = A \text{ and } \lambda' = \lambda)$

$\Rightarrow (X'A)Y = (\lambda X')Y$

$\Rightarrow X'(AY) = \lambda(X'Y)$

$\Rightarrow X'(\mu Y) = \lambda(X'Y)$ $\quad (\because AY = \mu Y)$

$\Rightarrow \mu(X'Y) = \lambda(X'Y)$

$\Rightarrow (\mu-\lambda)X'Y = 0$

$\Rightarrow X'Y = 0$ $\quad (\because \mu-\lambda \notin O_2)$

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