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On Non-homogeneous Ternary Cubic Equation

$$\mathbf{x^3 + y^3 + x + y = 2z(2z^2 - \alpha^2 + 1)}$$

A. Vijayasankar[1], Sharadha Kumar[2], M.A. Gopalan[3]

[1]Assistant Professor, Department of Mathematics, National College, Affiliated to Bharathidasan University, Trichy-620 001, Tamil Nadu, India.
email: avsankar70@yahoo.com

[2*]Research Scholar, Department of Mathematics, National College, Affiliated to Bharathidasan University ,Trichy-620 001, Tamil Nadu, India.
email: sharadhak12@gmail.com

[3]Professor, Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Trichy-620 002, Tamil Nadu, India.
email:mayilgopalan@gmail.com

ABSTRACT

This paper concerns with the problem of obtaining non-zero distinct integer solutions to the non-homogeneous cubic equation with three unknowns given $x^3 + y^3 + x + y = 2z(2z^2 - \alpha^2 + 1)$.A few interesting relations among the solutions are presented .Also ,a formula for generating sequence of integer solutions to the considered cubic equation based on its given solution is exhibited.

Key words: non-homogeneous cubic, ternary cubic, integer solutions , generation of solutions

INTRODUCTION

The cubic diophantine equations are rich in variety and offer an unlimited field for research [1,2]. In particular, refer [3-22] for a few problems on cubic equation with 3 unknowns for obtaining non-zero distinct integer solutions .This paper concerns with yet another non-homogeneous ternary cubic diophantine equation given by $x^3 + y^3 + x + y = 2z(2z^2 - \alpha^2 + 1)$ for determining its non-zero non- distinct integral solutions by employing the linear transformations. A few interesting relations among the solutions are presented. A general formula for generating sequence of integer solutions based on its given solution is exhibited.

Method of analysis

The non-homogeneous ternary cubic equation to be solved is

$$x^3 + y^3 + x + y = 2z(2z^2 - \alpha^2 + 1) \tag{1}$$

Introduction of the linear transformations

$$x = u + v, y = u - v, z = u, u \neq v \neq 0 \tag{2}$$

in (1) leads to

$$u^2 = 3v^2 + \alpha^2 \tag{3}$$

which is the well-known positive Pell equation .The general solution (v_{n+1}, u_{n+1}) to (3)

is given by

$$v_{n+1} = \frac{\alpha}{\sqrt{3}} g_n + \frac{\alpha}{2} f_n ,$$
$$u_{n+1} = \alpha f_n + \frac{\sqrt{3}}{2} \alpha g_n , n = -1,0,1,...$$

where

$$f_n = (2+\sqrt{3})^{n+1} + (2-\sqrt{3})^{n+1} , g_n = (2+\sqrt{3})^{n+1} - (2-\sqrt{3})^{n+1} ,$$

In view of (2),the general solution $(x_{n+1}, y_{n+1}, z_{n+1})$ to (1) is given by

$$\left.\begin{aligned} x_{n+1} &= \frac{3}{2}\alpha f_n + \frac{5\sqrt{3}}{6}\alpha g_n , \\ y_{n+1} &= \frac{1}{2}\alpha f_n + \frac{\sqrt{3}}{6}\alpha g_n , \\ z_{n+1} &= \alpha f_n + \frac{\sqrt{3}}{2}\alpha g_n , \end{aligned}\right\} n = -1,0,1,.... \tag{4}$$

A few numerical examples are presented in Table:1 below:

Table :1 Numerical examples

n	x_{n+1}	y_{n+1}	z_{n+1}
-1	3α	α	2α
0	11α	3α	7α
1	41α	11α	26α
2	153α	41α	97α
3	571α	153α	362α

From the above Table:1,the following results are observed:

(i) The values of x and y are both even or odd according as α is even or odd .

(ii) The values of z are even when α is even and alternatively even & odd when α is odd

(iii) $x_{n+1} = y_{n+2}$

(iv) $x_{n+1} + x_{n+3} = 4y_{n+3}$

(v) $z_{n+1} + z_{n+2} = 3x_{n+1}$

(vi) $z_{n+3} + z_{n+2} = 3y_{n+3}$

(vii) $y_{n+1} + x_{n+2} = 4y_{n+2}$

(viii) $y_{n+3} + y_{n+2} = 2z_{n+2}$

(ix) $z_{n+3} + 5z_{n+1} = 3(y_{n+3} + y_{n+1})$

(x) $x_{n+2} + x_{n+1} = y_{n+3} + y_{n+2}$

Each of the following expressions is a perfect square:

- $\alpha(8z_{2n+2} - 2z_{2n+3} + 2\alpha)$
- $\alpha(10z_{2n+2} - 6x_{2n+2} + 2\alpha)$
- $\alpha(18x_{2n+2} - 2z_{2n+4} + 2\alpha)$
- $\alpha(10z_{2n+2} - 6y_{2n+3} + 2\alpha)$
- $\alpha(5y_{2n+2} - x_{2n+2} + 2\alpha)$

Each of the following expressions is a cubical integer:

- $\alpha^2\left[5y_{3n+3} - x_{3n+3} + 3(5y_{n+1} - x_{n+1})\right]$
- $\alpha^2\left[10z_{3n+3} - 6y_{3n+4} + 3(10z_{n+1} - 6y_{n+2})\right]$
- $\alpha^2\left[18x_{3n+3} - 2z_{3n+5} + 3(18x_{n+1} - 2z_{n+3})\right]$
- $\alpha^2\left[10z_{3n+3} - 6x_{3n+3} + 3(10z_{n+1} - 6x_{n+1})\right]$

- $\alpha^2\left[8z_{3n+3}-2z_{3n+4}+3(8z_{n+1}-2z_{n+2})\right]$

Employing the linear combinations between the solutions of (1), one obtains integer solutions to special hyperbolas and parabolas :

Illustration 1:

The pairs of integers

$(X,Y)=(4z_{n+2}-14z_{n+1},8z_{n+1}-2z_{n+2})$,$(12x_{n+1}-18z_{n+1,}10z_{n+1}-6x_{n+1})$,
$(12y_{n+2}-18z_{n+1},10z_{n+1}-6y_{n+2})$,$(3x_{n+1}-9y_{n+1},5y_{n+1}-x_{n+1})$
satisfy the hyperbola $3\mathrm{Y}^2-\mathrm{X}^2=12\alpha^2$ correspondingly.

Illustration 2:

The pairs of integers
$(X,Y)=(4z_{n+2}-14z_{n+1},8z_{2n+2}-2z_{2n+3}+2\alpha)$,$(12x_{n+1}-18z_{n+1,}10z_{2n+2}-6x_{2n+2}+2\alpha)$,
$(12y_{n+2}-18z_{n+1},10z_{2n+21}-6y_{2n+3}+2\alpha)$,$(3x_{n+1}-9y_{n+1},5y_{2n+2}-x_{2n+2}+2\alpha)$
satisfy the hyperbola $3\alpha\mathrm{Y}-\mathrm{X}^2=12\alpha^2$ correspondingly.

Generation of Solutions:

The process of obtaining a formula for generating sequence of integer solutions based on the given solution is presented below:

Let $(u_0,v_{0,})$ be any given solution to (3).

Let (u_1,v_1) given by

$$u_1=2h-u_0,\ \ v_1=h+v_0 \tag{5}$$

be the 2^{nd} solution to (3). Using (5) in (3) and simplifying, one obtains

$$h=4u_0+6v_0$$

In view of (5), the values of u_1 and v_1 are written in the matrix form as

$$(u_1,v_1)^t=M(u_0,v_0)^t$$

where

$M=\begin{pmatrix}7 & 12\\ 4 & 7\end{pmatrix}$ and t is the transpose

The repetition of the above process leads to the n^{th} solutions u_n, v_n given by

$$(u_n, v_n)^t = M^n (u_0, v_0)^t \tag{6}$$

Now , if p, q are the distinct eigen values of M, then

$$p = 7 + 4\sqrt{3}, \; q = 7 - 4\sqrt{3}$$

We know that

$$M^n = \frac{p^n}{(p-q)}(M - qI) + \frac{q^n}{(q-p)}(M - pI), \; I = 2\times 2 \text{ Identity matrix}$$

and in view of (6),one obtains the values of u_n, v_n. Employing (2) ,the values of x_n, y_n, z_n satisfying (1) are given by

$$\left.\begin{aligned}
x_n &= \frac{1}{4\sqrt{3}}\left[(2\sqrt{3}(\alpha^n + \beta^n) + 4(\alpha^n - \beta^n))\,x_0 - 2(\alpha^n - \beta^n)\,y_0\right], \\
y_n &= \frac{1}{4\sqrt{3}}\left[(2\sqrt{3}(\alpha^n + \beta^n) - 4(\alpha^n - \beta^n))\,y_0 + 2(\alpha^n - \beta^n)\,x_0\right], \\
z_n &= \frac{1}{4}\left[(\alpha^n + \beta^n + \sqrt{3}(\alpha^n - \beta^n))\,x_0 + (\alpha^n + \beta^n - \sqrt{3}(\alpha^n - \beta^n))\,y_0\right]
\end{aligned}\right\} \tag{7}$$

In the above system (7), $x_0 = u_0 + v_0, y_0 = u_0 - v_0$

CONCLUSION

In this paper, we have made an attempt to determine non-zero distinct integer solutions to the non- homogeneous cubic equation with three unknowns given by $x^3 + y^3 + x + y = 2z(2z^2 - \alpha^2 + 1)$.. As the cubic equations are rich in variety, one may search for other forms of cubic equations with multiple variables to obtain their corresponding solutions .

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