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ON HOMOGENEOUS QUADRATIC EQUATION WITH THREE UNKNOWNS $(2\alpha+1)(x^2+y^2)-(2\alpha-1)xy=(8\alpha+4)z^2$

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ABSTRACT

The homogeneous quadratic equation with three unknowns represented by the Diophantine equation $(2\alpha+1)(x^2+y^2)-(2\alpha-1)xy=(8\alpha+4)z^2$ is analyzed for its patterns of non-zero distinct integral solutions through employing linear transformations..A general formula for generating sequence of integer solutions based on its given solution is exhibited.

KEYWORDS: homogenous quadratic equation, quadratic with three unknowns, integral solutions.

INTRODUCTION

Ternary quadratic equations are rich in variety [1-4, 17-19]. For an extensive review of sizable literature and various problems, one may refer [5-16]. In this communication, we consider yet another interesting homogeneous ternary quadratic equation

$(2\alpha+1)(x^2+y^2)-(2\alpha-1)xy=(8\alpha+4)z^2$ and obtain infinitely many non-trivial integral solutions..

METHOD OF ANALYSIS

The homogeneous quadratic equation with three unknowns to be solved for its distinct non-zero integral solutions is

$$(2\alpha+1)\left(x^2+y^2\right)-(2\alpha-1)xy=(8\alpha+4)z^2 \tag{1}$$

Substitution of the linear transformations

$$x=2X+(4\alpha-2)T,\ y=(8\alpha+4)T \tag{2}$$

in (1) leads to

$$z^2=X^2+(12\alpha^2+20\alpha+3)T^2 \tag{3}$$

Different ways of obtaining the patterns of integer solutions to (1) are illustrated below:

WAY: 1

It is seen that (3) is satisfied by

$$T=2pq\ ,\ X=(12\alpha^2+20\alpha+3)p^2-q^2$$

$$z=(12\alpha^2+20\alpha+3)p^2+q^2 \tag{4}$$

In view of (2),one has

$$x=(24\alpha^2+40\alpha+6)p^2-2q^2+(8\alpha-4)pq,\ y=(16\alpha+8)pq \tag{5}$$

Thus,(4) and (5) give the integer solutions to (1).

WAY: 2

Write (3) as the system of double equations as in Table : 1 below:

Table 1 : System of double equations

System	I	II
$z+X$	T^2	$(12\alpha^2+20\alpha+3)T$
$z-X$	$12\alpha^2+20\alpha+3$	T

Solving each of the above system of equations for z, X, T and using (2),the corresponding two sets of solution to (1) are as shown below:

Set:1

$$x = 4k^2 + 8k\alpha - 12\alpha^2 - 16\alpha - 4,$$
$$y = 16k\alpha + 8\alpha + 8k + 4,$$
$$z = 2k^2 + 2k + 2 + 6\alpha^2 + 10\alpha$$

Set:2

$$x = (12\alpha^2 + 24\alpha)T,$$
$$y = (8\alpha + 4)T,$$
$$z = (2 + 6\alpha^2 + 10\alpha)T$$

WAY: 3

(3) is written as

$$X^2 + (12\alpha^2 + 20\alpha + 3)T^2 = z^2 * 1 \qquad (6)$$

Assume

$$z = a^2 + (12\alpha^2 + 20\alpha + 3)b^2 \qquad (7)$$

Write 1 on the r.h.s. of (6) as

$$1 = \frac{(2\alpha - 1 + i\sqrt{12\alpha^2 + 20\alpha + 3})(2\alpha - 1 - i\sqrt{12\alpha^2 + 20\alpha + 3})}{(4\alpha + 2)^2} \qquad (8)$$

Using (7), (8) in (6) and applying the method of factorization, define

$$\left(X + i\sqrt{12\alpha^2 + 20\alpha + 3}\,T\right) = \frac{1}{4\alpha + 2}\left(2\alpha - 1 + i\sqrt{12\alpha^2 + 20\alpha + 3}\right)\left(a + i\sqrt{12\alpha^2 + 20\alpha + 3}\,b\right)^2$$

from which ,on equating the real and imaginary parts , one obtains the values of X and T. Substituting the above values of X & T in (2) and taking

$$a = (2\alpha + 1)A, b = (2\alpha + 1)B$$

the corresponding integer values of x, y and z satisfying (1) are as follows :

$$x=(2\alpha+1)\left[(4\alpha-2)(A^2-(12\alpha^2+20\alpha+3)B^2)-4(4\alpha^2+12\alpha+1)AB\right]$$

$$y=2(2\alpha+1)^2\left[(4\alpha-2)AB+A^2-(12\alpha^2+20\alpha+3)B^2\right]$$
$$z=(2\alpha+1)^2\left[A^2+(12\alpha^2+20\alpha+3)B^2\right]$$

Note :1

One may consider 1 on the r.h.s. of (6) as

$$1=\frac{(-(2\alpha-1)+i\sqrt{12\alpha^2+20\alpha+3})(-(2\alpha-1)-i\sqrt{12\alpha^2+20\alpha+3})}{(4\alpha+2)^2}$$

For this choice ,the corresponding integer values of x, y and z satisfying (1) are as follows :

$$x=-8(2\alpha+1)ab$$

$$y=2\left[a^2-(12\alpha^2+20\alpha+3)b^2-2(2\alpha-1)ab\right]$$

$$z=\left[a^2+(12\alpha^2+20\alpha+3)b^2\right]$$

WAY: 4

(3) is written as

$$z^2-(12\alpha^2+20\alpha+3)T^2=X^2*1 \tag{9}$$

Assume

$$X=a^2-(12\alpha^2+20\alpha+3)b^2 \tag{10}$$

Write 1 on the r.h.s. of (9) as

$$1=\frac{(4\alpha+2+\sqrt{12\alpha^2+20\alpha+3})(4\alpha+2-\sqrt{12\alpha^2+20\alpha+3})}{(2\alpha-1)^2} \tag{11}$$

Using (10), (11) in (9) and applying the method of factorization, define

$$\left(z+\sqrt{12\alpha^2+20\alpha+3}\,T\right)=\frac{1}{2\alpha-1}\left(4\alpha+2+\sqrt{12\alpha^2+20\alpha+3}\right)\left(a+\sqrt{12\alpha^2+20\alpha+3}\,b\right)^2$$

from which ,on equating the rational and irrational parts , one obtains the values of z and T. Substituting the above values of X & T in (2) and taking

$$a=(2\alpha-1)A, b=(2\alpha-1)B$$

the corresponding integer values of x, y and z satisfying (1) are as follows :

$$x=4(2\alpha-1)^2\left[(A^2+(4\alpha+2)AB\right]$$

$$y=4(4\alpha^2-1)\left[(8\alpha+4)AB+A^2+(12\alpha^2+20\alpha+3)B^2\right]$$

$$z=(2\alpha-1)\left[(4\alpha+2)(A^2+(12\alpha^2+20\alpha+3)B^2)+2(12\alpha^2+20\alpha+3)AB\right]$$

Note 2:

One may consider 1 on the R.H.S. of (9) as

$$1=\frac{(-(4\alpha+2)+\sqrt{12\alpha^2+20\alpha+3})(-(4\alpha+2)-\sqrt{12\alpha^2+20\alpha+3})}{(2\alpha-1)^2}$$

For this choice , the corresponding integer values of x, y and z satisfying (1) are as follows :

$$x=4(2\alpha-1)^2\left[(A^2-(4\alpha+2)AB\right]$$

$$y=4(4\alpha^2-1)\left[-(8\alpha+4)AB+A^2+(12\alpha^2+20\alpha+3)B^2\right]$$

$$z=(2\alpha-1)\left[-(4\alpha+2)(A^2+(12\alpha^2+20\alpha+3)B^2)+2(12\alpha^2+20\alpha+3)AB\right]$$

WAY: 5

Substitution of the linear transformations

$$x=u+v, y=u-v \tag{12}$$

in (1) leads to

$$(2\alpha+3)u^2+(6\alpha+1)v^2=(8\alpha+4)z^2$$

which is written in the form of ratio as

$$\frac{(2\alpha+3)(u+z)}{z+v}=\frac{(6\alpha+1)(z-v)}{u-z}=\frac{P}{Q}, Q\neq 0$$

Solving the above system of double equations through employing the method of cross-multiplication and using (12), the corresponding integer values of x, y and z satisfying (1) are as follows:

$$x = 2(8\alpha + 4)PQ$$

$$y = 2\left[P^2 - (12\alpha^2 + 20\alpha + 3)Q^2 + (4\alpha - 2)PQ\right]$$

$$z = \left[P^2 + (12\alpha^2 + 20\alpha + 3)Q^2\right]$$

CONCLUSION

In this paper, we have made an attempt to determine different patterns of non-zero distinct integer solutions to the homogeneous quadratic equation with three unknowns given by $(2\alpha+1)\left(x^2+y^2\right)-(2\alpha-1)xy=(8\alpha+4)z^2$. As the quadratic equations are rich in variety, one may search for other forms of quadratic equations with multiple variables to obtain their corresponding solutions .

References:

[1] Bert Miller, "Nasty Numbers", The Mathematics Teacher, Vol-73, No.9, Pp.649, 1980.

[2] Bhatia B.L and Supriya Mohanty, "Nasty Numbers and their Characterisation" Mathematical Education, Vol-II, No.1, Pp.34-37, July-September 1985.

[3] Carmichael R.D., The theory of numbers and Diophantine Analysis, NewYork, Dover, 1959.

[4] Dickson L.E., History of Theory of numbers, Vol.2: Diophantine Analysis, New York, Dover, 2005.

[5] Gopalan M.A., Manju Somnath, and Vanitha M., Integral Solutions of $kxy + m(x+y) = z^2$, Acta Ciencia Indica, Vol 33, No. 4,Pp.1287-1290, 2007.

[6] Gopalan M.A., Manju Somanath and Sangeetha V., On the Ternary Quadratic Equation $5(x^2+y^2)-9xy=19z^2$,IJIRSET, Vol 2, Issue 6, Pp.2008-2010, June 2013.

[7] Gopalan M.A., and Vijayashankar A., Integral points on the homogeneous cone $z^2 = 2x^2 + 8y^2$, IJIRSET, Vol 2(1), Pp.682-685, Jan 2013.

[8] Gopalan M.A., Vidhyalakshmi S., and Geetha V., Lattice points on the homogeneous cone $z^2 = 10x^2 - 6y^2$, IJESRT, Vol 2(2), Pp.775-779, Feb 2013.

[9] Gopalan M.A., Vidhyalakshmi S., and Premalatha E., On the Ternary quadratic Diophantine equation $x^2 + 3y^2 = 7z^2$, Diophantus.J.Math, 1(1), Pp.51-57, 2012.

[10] Gopalan M.A., Vidhyalakshmi S., and Kavitha A., Integral points on the homogeneous cone $z^2 = 2x^2 - 7y^2$, Diophantus.J.Math, 1(2), Pp.127-136, 2012.

[11] Gopalan M.A., and Sangeetha G., Observations on $y^2 = 3x^2 - 2z^2$, Antarctica J.Math., 9(4), Pp.359-362, 2012.

[12] Gopalan M.A., Manju Somanath and Sangeetha V., Observations on the Ternary Quadratic Diophantine Equation $y^2 = 3x^2 + z^2$, Bessel J.Math., 2(2), Pp.101-105,2012.

[13] Gopalan M.A., Vidhyalakshmi S., and Premalatha E., On the Ternary quadratic equation $x^2 + xy + y^2 = 12z^2$, Diophantus.J.Math, 1(2), Pp.69-76, 2012.

[14] Gopalan M.A., Vidhyalakshmi S., and Premalatha E., On the homogeneous quadratic equation with three unknowns $x^2 - xy + y^2 = (k^2 + 3)z^2$, Bulletin of Mathematics and Statistics Research, Vol 1(1), Pp.38-41, 2013.

[15] Meena.K, Gopalan M.A., Vidhyalakshmi S., and Thiruniraiselvi N., Observations on the quadratic equation $x^2 + 9y^2 = 50z^2$, International Journal of Applied Research, Vol 1(2), Pp.51-53, 2015.

[16] Anbuselvi R., and Shanmugavadivu S.A., On homogeneous Ternary quadratic Diophantine equation $z^2 = 45x^2 + y^2$, IJERA, 7(11), Pp.22-25, Nov 2017.

[17] Mordell L.J., Diophantine Equations, Academic press, London, 1969.

[18] Nigel,P.Smart, The Algorithmic Resolutions of Diophantine Equations, Cambridge University Press, London 1999.

[19] Telang, S.G., Number Theory, Tata Mc Graw-hill publishing company, New Delhi, 1996