International Journal of Research Publication and Reviews

Journal homepage: www.ijrpr.com ISSN 2582-7421

On The Homogeneous Cone $z^2 = 34x^2 + y^2$

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A B S T R A C T

The homogeneous ternary quadratic equation given by $z^2 = 34x^2 + y^2$ is analysed for its non-zero distinct integer solutions through different methods. A few interesting properties between the solutions are presented. Also, formulae for generating sequence of integer solutions based on the given solution are presented.

Keywords: Ternary quadratic, Integer solutions, Homogeneouscone

1. Introduction

It is well known that the quadratic Diophantine equations with three unknowns (homogeneous or non-homogeneous) are richin variety [1, 2]. In particular, the ternary quadratic Diophantine equations of the form $z^2 = Dx^2 + y^2$ are analysed for values of D=29,41,43,47, 53, 55, 61, 63, 67in [3-11]. In this communication, yet another interestinghomogeneousternary quadratic diophantine equation given by $z^2 = 34x^2 + y^2$ is analysed forits non-zero distinct integer solutions through different methods. A few interesting properties between the solutions are presented. Also, formulas for generating sequence of integer solutions based on the given solution are presented.

2. Methods of Analysis

The ternary quadratic equation to be solved for its integer solutions is

$$z^2 = 34x^2 + y^2 \qquad (1)$$

We present below different methods of solving (1):

Method: 1

(1) Is written in the form of ratio as

$$\frac{z+y}{34x} = \frac{x}{z-y} = \frac{\alpha}{\beta}, \beta \neq 0 \qquad (2)$$

which is equivalent to the system of double equations

$$34\alpha x - \beta y - \beta z = 0$$

$$\beta x + \alpha y - \alpha z = 0$$

Applying the method of cross-multiplication to the above system of equations,

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$$x = x(\alpha,\beta) = 2\alpha\beta$$
$$y = y(\alpha,\beta) = 34\alpha^2 - \beta^2$$
$$z = z(\alpha,\beta) = 34\alpha^2 + \beta^2$$

which satisfy (1)

Note: 1

It is observed that (1) may also be represented in the form of ratio as below:

(i) $\frac{z+y}{2x} = \frac{17x}{z-y} = \frac{\alpha}{\beta}, \beta \neq 0$

The corresponding solutions to (1) are given as:

$$x = 2\alpha\beta,\ y = 2\alpha^2 - 17\beta^2,\ z = 2\alpha^2 + 17\beta^2$$

(ii) $\frac{z+y}{17x} = \frac{2x}{z-y} = \frac{\alpha}{\beta}, \beta \neq 0$

The corresponding solutions to (1) are given as:

$$x = 2\alpha\beta,\ y = 17\alpha^2 - 2\beta^2,\ z = 17\alpha^2 + 2\beta^2$$

Method: 2

Is written as the system of double equation in Table 1 as follows:

Table: 1 System of Double Equations

System	I	II	III	IV
$z+y=$	$34x$	x^2	$17x^2$	$17x$
$z-y=$	x	34	2	$2x$

Solving each of the above system of double equations, the value of x, y & z satisfying (1) are obtained. For simplicity and brevity, in what follows, the integer solutions thus obtained are exhibited.

Solutions for system: I

$$x = 2k,\ y = 33k,\ z = 35k$$

Solutions for system: II

$$x = 2k,\ y = 2k^2 - 17,\ z = 2k^2 + 17$$

Solution for system: III

$$x = 2k,\ y = 34k^2 - 1,\ z = 34k^2 + 1$$

Solution for system: IV

$$x = 2k,\ y = 15k,\ z = 19k$$

Method: 3

(1) Is written as

$$y^2 + 34x^2 = z^2 = z^2 * 1 \quad (3)$$

Assume z as

$$z = a^2 + 34b^2 \quad (4)$$

Write 1 as

$$1 = \frac{(15 + 2i\sqrt{34})(15 - 2i\sqrt{34})}{19^2} \quad (5)$$

Using (4) & (5) in (3) and employing the method of factorization, consider

$$\left(y+i\sqrt{34}x\right)=\left(a+i\sqrt{34}b\right)^2\cdot\frac{15+2i\sqrt{34}}{19}$$

Equating real & imaginary parts, it is seen that

$$\left.\begin{aligned} y&=\frac{1}{19}\left[15(a^2-34b^2)-136ab\right]\\ x&=\frac{1}{19}\left[2(a^2-34b^2)+30ab\right]\end{aligned}\right\} \quad (6)$$

Since our interest is to find the integer solutions, replacing a by 19A & b by19B in (6) & (4), the corresponding integer solutions to (1) are given by

$$x=x(A,B)=19\left[2(A^2-34B^2)+30AB\right]$$

$$y=y(A,B)=19\left[15(A^2-34B^2-136AB\right]$$

$$z=z(A,B)=19^2\left[A^2+34B^2\right]$$

Note :2

It is worth to observe that, one may write 1 as follows:

$$1=\frac{\left[\left(34r^2-s^2\right)+i\sqrt{34}\cdot 2rs\right]\left[\left(34r^2-s^2\right)-i\sqrt{34}\cdot 2rs\right]}{\left(34r^2+s^2\right)^2}$$

$$1=\frac{\left[\left(2k^2-17\right)+i\sqrt{34}\cdot 2k\right]\left[\left(2k^2-17\right)-i\sqrt{34}\cdot 2k\right]}{\left(2k^2+17\right)^2}$$

Following the above procedure,one may obtain difference sets of integer solutions to (1).

Method: 4

(1) Is written as

$$z^2-34x^2=y^2=y^2*1 \quad (7)$$

Assume y as

$$y=a^2-34b^2 \quad (8)$$

Write 1 as

$$1=\frac{\left(19+2\sqrt{34}\right)\left(19-2\sqrt{34}\right)}{15^2} \quad (9)$$

Using (8) & (9) in (7) and employing the method of factorization, consider

$$\left(z+\sqrt{34}x\right)=(a+\sqrt{34}b)^2\cdot\frac{\left(19+2\sqrt{34}\right)}{15}$$

Equating rational and irrational parts, it is seen that,

$$\begin{aligned} x&=\frac{1}{15}\left(2(a^2+34b^2)+38ab\right)\\ z&=\frac{1}{15}\left(19(a^2+34b^2)+136ab\right)\end{aligned} \quad (10)$$

Since our interest to find the integer solution, replacing a by 15A & b by 15B in (10)& (8), the corresponding integer solutions to (1) are given by

$$x=x(A,B)=15\left[2(A^2+34B^2)+38AB\right]$$

$$y=y(A,B)=15^2\left[A^2-34B^2\right]$$

$$z=z(A,B)=15\left[19(A^2+34B^2)+136AB\right]$$

Note: 3

It is worth to observe that, one may write 1 as follows:

$$1=\frac{\left[\left(34r^2+s^2\right)+\sqrt{34}\cdot 2rs\right]\left[\left(34r^2+s^2\right)-\sqrt{34}\cdot 2rs\right]}{\left(34r^2-s^2\right)^2}$$

$$1=\frac{\left[\left(2k^2+17\right)+\sqrt{34}\cdot 2k\right]\left[\left(2k^2+17\right)-\sqrt{34}\cdot 2k\right]}{\left(2k^2-17\right)^2}$$

Following the above procedure,one may obtain difference sets of integer solutions to (1).

3. Generation of Solutions

Different formulas for generating sequence of integer solutions based on the given solution are presented below:

Let $\left(x_0, y_{0,} z_0\right)$ be any given solution to (1)

Formula: 1

Let $\left(x_{1,} y_1, z_1\right)$ given by

$$x_1 = 3x_0,\ y_1 = 3y_0 + h,\ z_1 = 2h - 3z_0 \qquad (11)$$

be the 2^{nd} solution to (1). Using (11) in (1) and simplifying, one obtains

$$h = 2y_0 + 4z_0$$

In view of (11), the values of y_1 and z_1 are written in the matrix form as

$$\left(y_1, z_1\right)^t = M\left(y_0, z_0\right)^t$$

where

$M = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$ and t is the transpose

The repetition of the above proses leads to the n^{th} solutions y_n, z_n givenby

$$\left(y_n, z_n\right)^t = M^n\left(y_0, z_0\right)^t$$

If α, β are the distinct eigen values of M, then

$$\alpha = 1,\ \beta = 9$$

We know that

$M^n = \frac{a^n}{(\alpha-\beta)}(M-\beta I) + \frac{\beta^n}{(\beta-\alpha)}(M-\alpha I),\ I = 2\times 2$ Identity matrix

Thus, the general formulas for integer solutions to (1) are given by

$$x_n = 3^n x_0$$

$$y_n = \left(\frac{9^n+1}{2}\right)y_0 + \left(\frac{9^n-1}{2}\right)z_0$$

$$z_n = \left(\frac{9^n-1}{2}\right)y_0 + \left(\frac{9^n+1}{2}\right)z_0$$

Formula: 2

Let $\left(x_1, y_1, z_1\right)$ given by

$$x_1 = h - 35x_0,\ y_1 = h - 35y_0,\ z_1 = 35z_0 \qquad (12)$$

be the 2^{nd} solution to (1). Using (12) in (1) and simplifying, one obtains

$$h = 68x_0 + 2y_0$$

In view of (12), the values of x_1 and y_1 are written in the matrix form as

$$\left(x_1, y_1\right)^t = M\left(x_0, y_0\right)^t$$

where

$M = \begin{pmatrix} 33 & 2 \\ 68 & -33 \end{pmatrix}$ and t is the transpose

The repetition of the above process leads to the n^{th} solutions x_n, y_n givenby

$$\left(x_n, y_n\right)^t = M^n\left(x_o, y_0\right)^t$$

If α, β arethe distinct eigen values of M, then

$$\alpha = 35, \beta = -35$$

Thus, the general formulas for integer solutions to (1) are given by

$$x_n = 35^{n-1}((34+(-1)^n)x_o + (1-(-1)^n)y_0)$$

$$y_n = 35^{n-1}((34(1-(-1)^n)x_o + (1+(-1)^n 34)y_0)$$

$$z_n = 35^n z_o$$

Formula: 3

Let $(x_1, y_1 z_1)$ given by

$$x_1 = x_0 + h, \ y_1 = y_0, \quad z_1 = 6h - z_0 \qquad (13)$$

be the 2^{nd} solution to (1). Using (13) in (1) and simplifying, one obtains

$$h = 34x_0 + 6z_0$$

In view of (13), the valuesof x_1 and z_1 are written in the matrix form as

$$(x_1, z_1)^t = M(x_0, z_0)^t$$

where

$$M = \begin{pmatrix} 35 & 6 \\ 204 & 35 \end{pmatrix}$$ and t is the transpose

The repetition of the above process leads to the n^{th} solutions x_n, z_n given by

$$(x_n, z_n)^t = M^n (x_0, z_0)^t$$

If α, β are the distinct eigen values of M, then

$$\alpha = 35 + 6\sqrt{34}, \ \beta = 35 - 6\sqrt{34}$$

Thus, the general formulas for integer solutions to (1) are given by

$$x_n = \left(\frac{\alpha^n + \beta^n}{2}\right)x_0 + \left[\frac{\alpha^n - \beta^n}{2\sqrt{34}}\right]z_0$$

$$y_n = y_0$$

$$z_n = \frac{17}{\sqrt{34}}(\alpha^n - \beta^n)x_0 + \left(\frac{\alpha^n + \beta^n}{2}\right)z_0$$

4. Conclusion

In this paper, an attempt has been made to obtain non-zero distinct integer solutions to the ternary quadratic Diophantine equation $z^2 = 34x^2 + y^2$ representing homogeneous cone. As there are varieties of cones, the readers may search for other forms of cones to obtain integer solutions for the corresponding cones.

REFFRENCE

[1]. L.E. Dickson, History of theory of Numbers, Vol. 2, Chelsea publishing Company, Newyork, 1952.

[2] L.J. Mordel, Diophantine Equations, Academic press, Newyork, 1969.

[3] Gopalan, M.A., Malika, S., Vidhyalakshmi, S., Integer solutions of $61x^2 + y^2 = z^2$, International Journal of Innovative science, Engineering and technology, Vol. 1, Issue 7, 271-273, September 2014.

[4] Meena K., Vidhyalakshmi S., Divya, S., Gopalan, M.A., Integer points on the cone $z^2 = 41x^2 + y^2$, Sch J., Eng. Tech., 2(2B), 301-304, 2014.

[5] Shanthi, J., Gopalan, M.A., Vidhyalakshmi, S., Integer solutions of the ternary, quadratic Diophantine equation $67X^2 + Y^2 = Z^2$, paper presented in International conference on Mathematical Methods and Computation, Jamal Mohammed College, Trichy, 2015

[6] Meena, K., Vidhyalakshmi, S., Divya, S., Gopalan M.A., On the ternary quadratic Diophantine equation $29x^2 + y^2 = z^2$, International journal of Engineering Research-online, Vol. 2., Issue.1., 67-71, 2014.

[7] Akila, G., Gopalan, M.A., Vidhyalakshmi, S., Integer solution of $43x^2 + y^2 = z^2$, International journal of engineering Research-online, Vol. 1., Issue.4., 70-74, 2013.

[8] Nancy, T., Gopalan, M.A., Vidhyalakshmi, S., On the ternary quadratic Diophantine equation $47x^2 + y^2 = z^2$, International journal of Engineering Research-online, Vol. 1., Issue.4., 51-55, 2013.

[9] Vidyalakshmi, S., Gopalan, M.A., Kiruthika, V., A search on the integer solution to ternary quadratic Diophantine equation $z^2 = 55x^2 + y^2$, International research journal of modernization in Engineering Technology and Science, Vol. 3., Issue.1, 1145-1150, 2021.

[10] Meena, K., Vidyalakshmi, S., Loganayagi, B., A search on the Integer solution to ternary quadratic Diophantine equation, $z^2 = 63x^2 + y^2$, International research journal of Education and Technology, Vol. 1, Issue.5, 107-116, 2021.

[11] Shanthi, J., Gopalan, M.A., Devisivasakthi, E., On the Homogeneous Cone $z^2 = 53x^2 + y^2$, International research Journal of Education and Technology, Vol. 1., Issue.4, 46-54, 2021.