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PI-Lattices a Classical of Bounded Commutative BCIK – Algebras

S Rethina Kumar

Assistant Professor, PG and Research Department Mathematics, Bishop Heber College(Affiliated to Bharathidasan University), Trichy, Tamil Nadu, India

A B S T R A C T

In this paper, Introduced BCIK – algebra and its properties, also we define the notion of PI - Lattices, as a generalization of finite positive implicative BCIK – algebra with bounded commutative BCIK–algebra. We investigate some results for PI-lattices being a new classical of BCIK–lattices. Specially, we prove that any Boolean lattice is a PI-lattice and we show that if X is a PI-lattice with bounded commutative, then X is an involutory BCIK-algebra if and only if X is a commutative BSIK-algebra. Finally, we prove that any PI-lattice with bounded commutative is a distributive BCIK-algebra.

Keywords: BCIK – algebra, PI-lattices, Involutroy BCIK-algebra, Bounded commutative BCIK-algebra

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1. Introduction

In 1966, Y. Imai and K. Iseki [1,2] defined BCK – algebra in this notion originated fromtwo different sources: one of them is based on the set theory the other is form the classical and non – classical propositional calculi. In [3]. Y.B. Jun and X.L. Xin applied the notion of derivation in ring and near – ring theory to BCI – algebras, and they also introduced a new concept called a derivation in BCI–algebras and its properties. We introduce combination BCK–algebra and BCI–algebra to define BCIK–algebra and its properties and also using Lattices theory to derived some basic definitions, theorems and its properties. George Boole's attempts to formalize propositional logic led to the concept of Boolean algebras.Investigating the axiomatic of Boolean algebras at the end of the nineteenth contury. S Charles, Peirce and Ernst Schroder found it useful to introduce the concept of a lattice. Dedekind also introduce modularity, a weakened form of distributivity. It was Garret Birkhoff's work in the mid-thirtics that started the general development of lattice theory. In a brilliant series of papers, he demonstrated the importance of lattice theory and showed that it provides a unifying framework for hitherto unrelated developments in many mathematical disciplines. In BCIK-algebra some important lattices such as bounded commutative BCIK-algebra, involutory BCIK-algebra and bounded implicative BCIK-algebra were proved [6, 8,9]. In order to extend the theory of bounded BCIK-algebras, we introduce the concept of PI-lattices and characterize their properties. We prove that the class of these lattices includes some currently know subclasses of BCIK-lattices such as bounded commutative BCIK-algebra (bounded commutative BCIK-lattices), finite positive implicative BCIK-algebra with bounded commutative and bounded implicative BCIK-algebras(Boolean lattices). We study the relation between involutory BCIK-algebras and PI-lattices and show that in PI-lattices with bounded commutative, commutative BCIK-algebra as and involutory BCIK-algebras coincide. Finally, we prove that any PI-lattice with bounded commutative is distributive.

2. Preliminaries

Definition 2.1.BCIK algebra

* *Corresponding author.*
E-mail address: rethinakumars66@gmail.com

Let X be a non-empty set with a binary operation * and a constant 0. Then (X,*,0) is called a BCIK Algebra, if it satisfies the following axioms for all x,y,z Є X:

(BCIK-1) x*y = 0, y*x = 0, z*x = 0 this imply that x = y = z.

(BCIK-2) ((x*y)*(y*z))*(z*x) = 0.

(BCIK-3) (x*(x*y))*y = 0.

(BCIK-4) x*x = 0, y*y = 0, z*z = 0.

(BCIK-5) 0*x =0, 0*y = 0, 0*z = 0.

For all x,y,z Є X. An inequality ≤ is a partially ordered set on X can be defined x ≤ y if and only if (x*y)*(y*z) = 0.

Properties 2.2.[5] I any BCIK – Algebra X, the following properties hold for all x,y,z Є X:

(1) 0 Є X.
(2) x*0 = x.
(3) x*0 = 0 implies x = 0.
(4) 0*(x*y) = (0*x)*(0*y).
(5) X*y = 0 implies x = y.
(6) X*(0*y) = y*(0*x).
(7) 0*(0*x) = x.
(8) x*y Є X and x Є X imply y Є X.
(9) (x*y)*z = (x*z)*y
(10) x*(x*(x*y)) = x*y.
(11) (x*y)*(y*z) = x*y.
(12) 0 ≤ x ≤ y for all x,y Є X.
(13) x ≤ y implies x*z ≤ y*z and z*y ≤ z*x.
(14) x*y ≤ x.
(15) x*y ≤ z ⇔ x*z ≤ y for all x,y,z Є X
(16) x*a = x*b implies a = b where a and b are any natural numbers (i.e)., a,b Є N
(17) a*x = b*x implies a = b.
(18) a*(a*x) = x.

Definition 2.3.[5]Let X be a BCIK – algebra. Then, for all x, y, z Є X:

(1) X is called a positive implicative BCIK – algebra if (x*y)*z = (x*z)*(y*z).
(2) X is called an implicative BCIK – algebra if x*(y*x) = x.
(3) X is called a commutative BCIK – algebra if x*(x*y) = y*(y*x).
(4) X is called bounded BCIK – algebra, if there exists the greatest element 1 of X, and for any xЄ X, 1*x is denoted by GG_x,
(5) X is called involutory BCIK – algebra, if for all x Є X, GG_x = x.

Definition 2.4. [5] Let X be a bounded BCIK-algebra. Then for all x, y Є X:

(1) G1 = 0 and G0 = 1,
(2) $GG_x \leq$ x that $GG_x = G(G_x)$,
(3) $G_x * G_y \leq$ y*x,
(4) y ≤ x implies $G_x \leq G_y$,
(5) $G_{x*y} = G_{y*x}$
(6) $GGG_x = G_x$.

Theorem 2.5. [5] Let X be a bounded BCIK-algebra. Then for any x, y Є X, the following hold:

(1) X is involutory,
(2) x*y = $G_y * G_x$,
(3) x*G_y = y * G_x,
(4) x ≤ G_y implies y ≤ G_x.

Theorem 2.6.[5] Every implicative BCIK-algebra is a commutative and positive implicative BCIK-algebra.

Definition 2.7. [5] Let Xbe a BCIK-algebra. Then:

(1) X is said to have bounded commutative, if for any x, y Є X, the set A(x,y) = {t Є X : t*x ≤ y} has the greatest element which is denoted by x o y,
(2) (X,*,≤) is called a BCIK-lattices, if (X,≤) is a lattice, where ≤ is the partial BCIK-order on X, which has been introduced in Definition 2.1.

Definition 2.8. [5] Let X be a BCIK-algebra with bounded commutative. Then for all x, y, z Є X:

(1) y ≤ x o (y*x),
(2) (x o z)*(y o z) ≤ x*y,
(3) (x*y)*z = x*(y o z),
(4) If x ≤ y, then x o z ≤ y o z,
(5) z*x ≤ y ⇔ z ≤ x o y.

Theorem 2.9. [5] Let X be a BCIK-algebra with condition bounded commutative. Then, for all x, y, z Є X, the following are equivalent:

(1) X is a positive implicative,
(2) x ≤ y implies x o y = y,
(3) x o x = x,
(4) (x o y) * z = (x*z) o (y*z),
(5) x o y = x o (y*x).

Theorem 2.10. [5] Let X be a BCIK-algebra.

(1) If X is a finite positive implicative BCIK-algebra with bounded and commutative the (X,≤) is a distributive lattice,
(2) If X is a BCIK-algebra with bounded and commutative, then X is positive implicative if and only if (X,≤) is an upper semilattice with x ∨ y = x o y, for any x, y Є X,
(3) If X is bounded commutative BCIK-algebra, then BCIK-lattice (X,≤) is a distributive lattice, where x ∧ y = y*(y*x) and x ∨ y= $G(G_x \wedge G_y)$.

Theorem 2.11.[5] Let X be an involutory BCIK-algebra, Then the following are equivalent:

(1) (X,≤) isa lower semilattice,
(2) (X,≤) is an upper semi lattice,
(3) (X,≤) is a lattice.

Theorem 2.12. [5] Let X be a bounded BCIK-algebra. Then:

(1) every commutative BCIK-algebra is an involutory BCIK-algebra.
(2) Any implicative BCIK-algebra is a Boolean lattice (a complemented distributive lattice).

Theorem 2.13.[5] Let X be a BCK-algebra, Then, for all x, y, z Є X, the following are equivalent:

(1) X is commutative,
(2) x*y = x*(y*(y*x)),
(3) x*(x*y) = y*(y*(x*(x*y))),
(4) x≤ y implics x = y*(y*x).

3. PI-lattices

In this section, we define the notion of PI-lattice in BCIK-algebra, which is a new class of bounded BCIK-algebras.

Definition 3.1. Let (X,*,≤) be a BCIK-lattice. Then (X,*,≤) is called a PI-lattice, if

(z*x)*(y*x) = z*(x ∨ y) for all x, y, z Є X.

Note. In what follows, we show that the class of PI-lattices includes the finite positive implicative BCIK-algebra with bounded commutative BCIK-algebra that are two important classes of BCIK-lattices. Hence this class of BCIK-lattices is called PI-lattice.

Example 3.2.

Let X = {0, a, b, 1} be a chain, where 0 ≤ a ≤ b ≤ 1, and the operation * on Xis defined as follows:
It is easy to check that (X,*,≤) is a PI-lattice.

*	0	a	B	1
0	0	0	0	0
A	a	0	0	0
B	b	A	0	0
1	1	1	1	0

It is easy to check that (X, *,≤) is a PI-lattice.

Let X be an interval [0, 1] of real number and binary operation * on X is defined as a follows:

x*y = 0 if x ≤ y

1 if x≥ y

Then it is easy to check that (X,*,≤) is a PI-lattice, where x ∧ y = min{x, y} and xΛ y = max{x, y}.

The following example shows that a BCIK-lattice is not a PI-lattice, in general.

Example 3.3. Let X = {0, a, b, 1} be a chain, that is 0 ≤ a ≤ b ≤ 1, and the operation * on X is defined as follows:

*	0	a	B	1
0	0	0	0	0
A	a	0	0	0
B	b	b	0	0
1	1	b	A	0

Then it is easy to check that (X,*,≤) is a BCIK-lattices but it is not a PI-lattice, since

(1*a)*(b*a) = b*b = 0 ≠ a = 1*b 1*(b∨a).

Theorem 3.4. Let X be a bounded BCIK-algebra. Then:

Every commutative BCIK-algebra is a PI-lattice.

Every finite positive implicative BCIK-algebra with bounded commutative is a PI-algebra

Proof:

Let X be a commutative BCIK-algebra. By Theorem 2.10 (3), X is a BCIK-lattice and by Theorem 2.12 (1), X is an involutory BCIK-algebra. Hence for any x, y, z∈ X, we have:

(x*x)*(y*x) = (G_x*G_z)*(G_x*G_y). by Theorem 2.5 (2)

= (G_x*(G_x*G_y))*G_z, by Definition 20.2 (1)

= (G_xΛG_y)*G_z, by Theorem 2.10 (3)

= GG(G_xΛG_y)*G_z, by Definition 2.3 (4)

= z*G(G_xΛG_y), by Theorem 2.5 (2)

= z*(x⋁y). by Theorem 2.10 (3)

Therefore, X is a PI-lattice.

Let X be a finite positive implicative BCIK-algebra with bounded commutative. Then by Theorem 2.10 (1), X is a BCIK-lattice and so for any x, y, z ∈ X,

(z*x)*(y*x) = z*(x o (y*x)), by Definition 2.8 (3)

= z*(x o y), by Theorem 2.9(5)

= z*(x⋁y). by Theorem 2.10 (3)

Hence X is a PI-lattice

The following example shows the converse of Theorem 3.4does not hold, in general.

Example 3.5. Let X = {0, a, b, 1} be a chain, where 0 ≤ a ≤ b ≤ 1, and the operation * on X is defined as follows:

*	0	a	B	1
0	0	0	0	0
A	a	0	0	0
B	b	a	0	0
1	1	1	1	0

It is easy to check that X is a PI-lattice, but it is not a positive implicative BCIK-algebra, since (b*a)*(a*a) = a*0 = a ≠ a*a = (b*a)*a. Also X is not a commutative BCIK-algebra. Since 1*(1*a) = 1*1 = 0 ≠ a = a*0 = a*(a*1).

Corollary 3.6. Any bounded implicative BCIK-algebra(Boolean lattice) is a PI-lattice.

Proof: By Theorem 2.6, 2.12 (1) and 3.4 (1), the proof is clear.

Proposition 3.7. Let X be a PI-lattice. Then for all x, y, z ∈ X. Since x ≤ x ⋁ y and y ≤ x ⋁ y, by Proposition 2.2 (2), we have (x*z) ≤ (x⋁y)*z. On the other hand, since X is a PI-lattice, By Definition 3.1 used repeatedly, we have:

[(x⋁y)*z]*[(x*y)⋁(y*z)] = [((x⋁y)*z)*(x*z)]*[(y*z)*(x*z)]

= ((x⋁y)*(x⋁z))*((y*z)*(x*z))

= ((x⋁y)*(x⋁z))*(y*(x⋁z))

= (x⋁y)*((x⋁z)⋁y)

= 0

Hence, (x⋁y)*z ≤ (x*z)⋁(y*z).

Therefore (x⋁y)*z = (x*z)⋁(y*z).

Theorem 3.8. Let X be abounded BCIK-algebra.

If X is a finite PI-lattice, then X is a BCIK-algebra with bounded commutative

If X is a BCIK-algebra with bounded commutative, then x ⋁y = x o (y*x), for x, y Є X.

Proof: (1) Suppose that (X, *, ≤) is a finite PI-lattice and x, y Є X. Then the set A(x,Y) = {t: t*x≤y} is a finite subset of X. Let A(x,y) = {t_i /iЄ I}, where I = {1,2,….n}. Since X is a lattice, there exists z Є X such that ⋁{t_i : t_i* x≤y,iЄI} = z. Hence, by Definition 3.7 (2).

z*x = {⋁t_i}*x = ⋁(t_i*x)≤⋁y = y for all I Є I

Therefore, zЄA(x, y). It is clear that for every wЄA(x,y), w≤z, and so z is the greatest element of the set A(x,y), which is defined by x o y. Hence every finite PI-lattice is a BCIK-algebra with bounded commutative

Suppose that X is a PI-algebra with bounded commutative. Then for x, y ЄX, the set A(x,y*x) has the greatest element. Now, since

((x ⋁y)*x)*(y*x) = (x⋁y)*(x⋁y) = 0

We have (x⋁y)*x≤y*x. Hence (x⋁y) Є A(x, y*x), then t*x≤y*x, and so (t*x)*(y*x) = 0 and so t ≤x⋁y.

Hence. X ⋁y is the greatest element of A(x, y*x) that is xo(y*x) = x⋁y.

The following example shows that the converse of Theorem 3.8, is not correct in general.

Example 3.9. Let X = {0, a, b, 1} be a chain, where 0 ≤ a ≤ b ≤ 1, and the operation * on X is defined as follows:

*	0	a	b	1
0	0	0	0	0
A	a	0	0	0
B	b	a	0	0
1	1	a	a	0

*	0	A	b	1
0	0	A	b	1
A	A	1	1	1
A	a	1	1	1
1	1	1	1	1

Then X is a BCIK-algebra with bounded commutative, but it is not a PI-lattice.

Since

(1*a)*(b*a) = a*a = 0 ≠ a = 1*b = 1*(b⋁a)

Also we have a o(b*a) = 1≠a⋁b = b.

Theorem 3.10. Let X be a BCIK-algebra.Then:

x*(y⋀z) = (x*y)⋁(x*z), for x, y, z Є X,

if X is with bounded commutative, then for x, y, z Є X, xo(y⋀z) = (x o y)⋀(xoz).

Proof:

Suppose that X is a BCIK-lattice and x, y, z Є X. Since y⋀z≤y and y⋀z≤z, by Proposition 2.2 (2), we have x*y ≤ x*(y⋀z). Hence (x*y)⋁(x*z)≤ x*(y⋀z)

Now it remains to prove that x*(y⋀z)≤ (x*y)⋁(x*z). Since (x*y)≤ (x*y)⋁(x*z), by Proposition 2.2(2), and BCIK-algebra, we have x*((x*y)⋁(x*z))≤ x*(x*z)≤z. Hence x*((x*z)⋁(y*z))≤(y⋀z). By Proposition 2.2 (4), we conclude that

x*(y⋀z)≤ (x*y)⋁(x*z)

Therefore

x*(y⋀z)= (x*y)⋁(x*z)

Suppose that X is a BCIK-lattice with bounded commutative and x, y, z Є X. Since y⋀z≤y and y⋀z≤z, by Proposition 2.8 (4), we have xo (y⋀z)≤ (xo y) and xo (y⋀z)≤ (xo z), and so

xo (y⋀z)≤ (xo y)⋀(xo z)

Now it remains to prove that (xo y)⋀(xo z)≤xo(y⋀z)

Therefore

(xo y)⋀(xo z)≤xo(y⋀z)

Theorem 3.11. Let X be a PI-lattice with bounded commutative. Then the following are equivalent:

X is an involutory BCIK-algebra,

X is a commutative BCIK-algebra.

Proof: (1) ⇒ (2) Let X be an involutory BCIK-algebra and y≤x. Since G_x*G_y = 0 and G_x*G_y = (1*x)*(1*y)≤y*x = 0, it follows that (G_x*G_y)⋁ (G_y*G_y) = 0. Hence

$(G_x \text{ o } (x*y))*G_y = (G_x\text{o } (G_y*G_x))*G_y$, by Theorem 3.8

$= (G_x \vee G_y)*G_y$, by Theorm 3.8

$= (G_x*G_y)\vee(G_y \vee G_y)$, by Definition 3.7 (2)

$= 0$

Hence,

$Y*(x*(x*y)) = y*(GG_x(x*y))$

$= y*((1*G_x)*(x*Y))$

$= y*(1*(G_x\text{o } (x*y)))$, by Definition 2.8(3)

$= GG_y*G(G_x\text{o } (x*y))$

$\leq (G_x\text{o } (x*y))*G_y$

$= 0$

Therefore, $y \leq x*(x*y)$. On the other hand. By BCIK-algebra, we have $x*(x*y) \leq y$. Consequently, $y = x*(x*y)$. Hence by Theorem 2.13. $(X, *, \leq)$ is a commutative BCIK-algebra.

(2)⇒(1). The proof holds by Theorem 2.12(1).

Theorem 3.12. Let x is a PI-lattice with bounded commutative. Then X is a distributive lattice.

Proof: Suppose that X is a PI-lattice with bounded commutative. It is clear that $x\wedge (y\vee z) \leq (x\vee y)\vee(x\vee z)$, for any x, y, z Є X. Now it remains to show that $x\vee (y\wedge z) \leq (x\vee y)\wedge(x\vee z)$. Since X satisfies the bounded commutative, we have;

$x\vee (y\wedge z) = (y\wedge z) \vee x = (y\wedge z)\text{o}(x*(y\wedge z))$, by Theorem 3.8 (2)

$= (x*(y\wedge z)) \text{ o } (y\wedge z)$

$= ((x*(y\wedge z))\text{o } y) \wedge ((x*(y\wedge z))\text{o } z)$, by Definition 3.7(1)

$\geq ((x*y) \text{ o } y) \wedge ((x*z)\text{o } z)$

$= (x\vee y)\wedge (x\vee z)$, by Theorem 3.8(2)

Therefore, X is a distributive lattice.

4. Conclusion

It is well-known that the concept of the lattices theory has an important role in investigating the structure of a logical system. Also a Boolean lattice, that is a complemented distributive lattice, has many applications in the computer science. In order to extend the concept of BCIK-algebras, we have proposed the concepts of PI-lattices. Then we have established the relationships between these lattices and other currently known lattices in bounded BCIK-algebras and proved that PI-lattices are distributive.

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