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Solvable Pseudo Commutators BCIK-Algebras

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A B S T R A C T

In this paper, Introduced BCIK – algebra and its properties. We investigate the notion of derived sub-algebras and solvable BCIK-algebras are introduced and some properties are given. We introduce the notion of commutators in BCIK-algebra and also discuss their properties. It has been found that the sub-algebra, isomorphic image and inverse image of a solvable BCIK-algebra are still solvable BCIK-algebra.

Keywords: BCIK-algebra, BCIK-ideal, commutators, derived sub-algebra, solvable Pseudo commutators BCIK-algebra.

1. Introduction

In 1966, Y. Imai and K. Iseki [1,2] defined BCK – algebra in this notion originated from two different sources: one of them is based on the set theory the other is form the classical and non – classical propositional calculi. In [3]. Y.B. Jun and X.L. Xin applied the notion of derivation in ring and near – ring theory to BCI – algebras, and they also introduced a new concept called a derivation in BCI–algebras and its properties. We introduce combination BCK–algebra and BCI–algebra to define BCIK–algebra and its properties and also using Lattices theory to derived the some basic definitions, an algebra of type (1,0), also known as BCIK-algebra, as a generalization the notation of algebra sets with the subtraction set with the only a fundamental, non-nullary operation and the notion of implication algebra [13,14] on the other hand. This notion is derived using two different methodologies, one of which is based on set theory and the other on classical and non-classical propositional calculi.

2. Preliminaries

Definition 2.1 BCIK algebra

Let X be a non-empty set with a binary operation * and a constant 0. Then (X, * ,0) is called a BCIK Algebra, if it satisfies the following axioms for all x, y, z Є X:

(BCIK-1) x*y = 0, y*x = 0, z*x = 0 this imply that x = y = z.

(BCIK-2) ((x*y) * (y*z)) * (z*x) = 0.

(BCIK-3) (x*(x*y)) * y = 0.

(BCIK-4) x*x = 0, y*y = 0, z*z = 0.

(BCIK-5) 0*x =0, 0*y = 0, 0*z = 0.

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For all x, y, z Є X. An inequality ≤ is a partially ordered set on X can be defined x ≤ y if and only if
(x*y) * (y*z) = 0.

Properties 2.2. [5] I any BCIK – Algebra X, the following properties hold for all x, y, z Є X:

(1) 0 Є X.
(2) x*0 = x.
(3) x*0 = 0 implies x = 0.
(4) 0*(x*y) = (0*x) * (0*y).
(5) X*y = 0 implies x = y.
(6) X*(0*y) = y*(0*x).
(7) 0*(0*x) = x.
(8) x*y Є X and x Є X imply y Є X.
(9) (x*y) * z = (x*z) * y
(10) x*(x*(x*y)) = x*y.
(11) (x*y) *(y*z) = x*y.
(12) 0 ≤ x ≤ y for all x, y Є X.
(13) x ≤ y implies x*z ≤ y*z and z*y ≤ z*x.
(14) x*y ≤ x.
(15) x*y ≤ z ⇔ x*z ≤ y for all x, y, z Є X
(16) x*a = x*b implies a = b where a and b are any natural numbers (i. e)., a, b Є N
(17) a*x = b*x implies a = b.
(18) a*(a*x) = x.

Definition 2.3. [4, 5, 6, 7] Let X be a BCIK – algebra. Then, for all x, y, z Є X:

(1) X is called a positive implicative BCIK – algebra if (x*y) * z = (x*z) * (y*z).
(2) X is called an implicative BCIK – algebra if x*(y*x) = x.
(3) X is called a commutative BCIK – algebra if x*(x*y) = y*(y*x).
(4) X is called bounded BCIK – algebra, if there exists the greatest element 1 of X, and for any x Є X, 1*x is denoted by GG_x,
(5) X is called involutory BCIK – algebra, if for all x Є X, GG_x = x.

Definition 2.4. [5, 7] Let X be a bounded BCIK-algebra. Then for all x, y Є X:

(1) G1 = 0 and G0 = 1,
(2) $GG_x \le x$ that $GG_x = G(G_x)$,
(3) $G_x * G_y \le y*x$,
(4) y ≤ x implies $G_x \le G_y$,
(5) $G_{x*y} = G_{y*x}$
(6) $GGG_x = G_x$.

Theorem 2.5.[8] Let X be a bounded BCIK-algebra. Then for any x, y Є X, the following hold:

(1) X is involutory,
(2) $x*y = G_y * G_x$,
(3) $x*G_y = y * G_x$,
(4) $x \le G_y$ implies $y \le G_x$.

Theorem 2.6.[5] Every implicative BCIK-algebra is a commutative and positive implicative BCIK-algebra.

Definition 2.7. [10,11] Let X be a BCIK-algebra. Then:

(1) X is said to have bounded commutative, if for any x, y Є X, the set A(x,y) = {t Є X : t*x ≤ y} has the greatest element which is denoted by x o y,
(2) (X, *, ≤) is called a BCIK-lattices, if (X,≤) is a lattice, where ≤ is the partial BCIK-order on X, which has been introduced in Definition 2.1.

Definition 2.8.[11] Let X be a BCIK-algebra with bounded commutative. Then for all x, y, z Є X:

(1) y ≤ x o (y*x),
(2) (x o z) * (y o z) ≤ x*y,
(3) (x*y) * z = x*(y o z),
(4) If x ≤ y, then x o z ≤ y o z,
(5) z*x ≤ y ⇔ z ≤ x o y.

Theorem 2.9. [12] Let X be a BCIK-algebra with condition bounded commutative. Then, for all x, y, z Є X, the following are equivalent:

(1) X is a positive implicative,
(2) $x \leq y$ implies x o y = y,
(3) x o x = x,
(4) (x o y) * z = (x*z) o (y*z),
(5) x o y = x o (y*x).

Theorem 2.10. [8, 9, 10] Let X be a BCIK-algebra.

(1) If X is a finite positive implicative BCIK-algebra with bounded and commutative the (X, ≤) is a distributive lattice,
(2) If X is a BCIK-algebra with bounded and commutative, then X is positive implicative if and only if (X, ≤) is an upper semi lattice with $x \vee y$ = x o y, for any x, y Є X,
(3) If X is bounded commutative BCIK-algebra, then BCIK-lattice (X, ≤) is a distributive lattice, where $x \wedge y = y*(y*x)$ and $x \vee y= G(G_x \wedge G_y)$.

Definition 2.11. An algebra (X,*,0) 0f type (2,0) is called a BCIK-algebra, if it satisfies the following axiom: for all x, y, z Є X,

(1) ((x*y)*(x*z))*(z*Y) = 0
(2) (x*(x*y))*y =0,
(3) X*x=0,
(4) 0*x=0,
(5) X*y=y*x=0 implies x=y

Definition 2.12. A nonempty subset S of a BCIK-algebra X is called a BCIK-sub-algebra of X, if x*y Є S. Moreover, a nonempty subset I of a BCIK-algebra X is called a BCIK-ideal if [15]

0 Є I
X*y Є I and y Є I imply x Є I for all x,y Є X.

Definition 2.13. A mapping f:X→Y is called a homomorphism from X to Y if for any x, y Є X, f(x*y) = f(x).f(y) holds.

Definition 2.14. In any commutative BCIK-algebra, the following statements holds:

(1) x⋀ x = x ⋁ x = x
(2) x⋁ 0 = 0 ⋁ x = x ⋀ 1 = 1 ⋀ x = x
(3) x ⋀ y = y ⋀ x
(4) x ⋁ y = y ⋁ x
(5) x ⋁ 1 = 1 ⋁ x = 1
(6) 0 ⋀ x = x ⋀ 0 = 0
(7) $GG_x = x$

For any fixed elements a≤b of a BCIK-algebra X, the set[a,b] = {xЄ X: a≤x≤b} = { xЄX: ax=xb=0}, is called the segment of X[16]. Note that the segment [o,b] = {x Є X: x≤b} = {x Є X:x*b = 0} is called initial. Let I be an ideal of a BCIK-algebra X. Define an equivalence relation ≈ on X by x ≈ y if and only if x*y, y*x Є I.
Definition 2.15. Let C_x denote the class of x Є X. Then X/I is called the quotient BCIK-algebra of X determined by I with $C_x*C_y = C_{x*y}$ and $C_x \leq C_y$ if and only if x≤y, the notion of commutators in BCIK-algebras is considered and some related are obtained[17].

3. Pseudo Commutators in BCIK-Algebras

From now on, for simply in this section X is a BCIK-algebra, unless otherwise is stated.

Definition 3.1. [17] Let $x_1,x_2,\ldots,x_n$ be elements of X. Then the elements $(x_1 \wedge x_2)*(x_2 \wedge x_1)$ of X is called Pseudo commutators of x_1 and x_2 of weight 2 and denoted by $\{x_1,x_2\}$. i.e.,

$$[x_1,x_2] = (x_1 \wedge x_2)*(x_2 \wedge x_1)$$

In general, the element $[x_1,x_2,\ldots,x_n] = [[x_1,x_2,\ldots,x_{n-1}],x_n]$ is a commutator of weight $n \geq 2$, where by convention $[x_1] = x_1$. A useful shorthand notation is $[x,y]_n = [x,y,\ldots,y]$ n times

Example 3.2. [17] Let X = {0,1,2,3,4}. Define * by the following table

4. *	5. 0	6. 1	7. 2	8. 3	9. 4
10. 0	11. 0	12. 0	13. 0	14. 0	15. 0
16. 1	17. 1	18. 0	19. 1	20. 0	21. 0
22. 2	23. 2	24. 0	25. 0	26. 0	27. 0
28. 3	29. 3	30. 3	31. 3	32. 0	33. 0
34. 4	35. 4	36. 3	37. 4	38. 1	39. 0

Then (X,*,0) is a bounded positive implicative BCI-algebra with the largest element 4, we have [2,4] = 0 ≠2 = [4,2]. By Definition 3.1. and Example 3.2., in the case there [x, y] ≠ [y, x] in BCIK-algebra, so this definition of commutators is not the commutators in the sense of usual commutators of group theory. Thus the commutators defined in this paper are essentially directed commutators.

Proposition 3.3.[17] The properties of commutators in BCIK-algebra

For any x, y∈ X

(1) If x≤y, then [x,y] = 0.
(2) [x,0] = [0,x] = [x,x] = 0.
(3) [x,y]*y = 0.
(4) [x,y]*y = 0.

Definition 3.4. [17] Let $X_1,X_2,\ldots,X_n$ be nonempty subset of X. Define the pseudo commutator of subset of X_1 and X_2 to be $[X_1,X_2] = \{[X_1,X_2]: x_1 \in X_1, x_2 \in X_2]$. More generally, $X_1,X_2,\ldots,X_n = [[X_1,X_2,\ldots,X_{n-1}],X_n]$ where n≥2. Furthermore, the subset [X,X] = {[a,b]/a,b ∈ X} of X is called the derived subset of X. In the general case , the pseudo commutators of X are not a sub-algebra of X.

Example 3.5. Let X = {0,1,2,3,4} and let the * operation be defined by the following table

40. *	41. 0	42. 1	43. 2	44. 3	45. 4
46. 0	47. 0	48. 0	49. 0	50. 0	51. 0
52. 1	53. 1	54. 0	55. 0	56. 0	57. 0
58. 2	59. 2	60. 2	61. 0	62. 2	63. 0
64. 3	65. 3	66. 1	67. 1	68. 0	69. 1
70. 4	71. 4	72. 4	73. 4	74. 4	75. 0

Then (X,*,0) is a BCIK-algebra. A = {4) and B = {2) are two subsets of X. However, [A,B] = {[4,2]} = {2}, therefore 2 ∈ [A,B] and 2*2 = 0 is not member of [A,B], i.e [A,B] is not sub-algebra of X.

Definition 3.5. $[X,X] = \{\prod[a_i,b_i]:a_i,b_i \in X\}$, where ∏ product represents a finite number of commutators. [X,X] is called the derived sub-algebras of X and is denoted by X'. $X' = [X,X] = \{\prod[a_i,b_i]:a_i,b_i \in X\}$, $X'' = [X',X'] = \{\prod[a_i,b_i]:a_i,b_i \in X'\}$, $X^{(n)} = [X^{(n-1)},X^{(n-1)}] = \{\prod[a_i,b_i]:a_i,b_i \in X^{(n-1)}\}$.
Generally, $[X_1,X_2] \neq [X_2,X_1]$, we write $[X,Y]_n$ for [X,Y,….., Y] n times. For any x ∈ X, we have [x,0] = [0,x] = [x,x] = 0. For any two nonempty sub-algebra A,B of X, so 0 ∈ [A,B], that is 0 ∈ X'.

Example 3.6. Let X = [0,1]. Define * on X by

x*y = 0 if x≤y
x otherwise

Then (X,*,0) is a bounded BCIK-algebra. Let $X_1 = [0,1], X_2=[0,1/2], X_3=[0,1/3],\ldots, X_n =[0,1/n]$.Then $[X_1,X_2] = [0,1/2],[X_2,X_1] = [0,1/2]$, $[X_1,X_2,\ldots,X_n] = [0,1/n]$. Also for any n≥1, there is $X^{(n)} = 0$.
The notions of pseudo commutators in BCIK-algebra and derived sub-algebra $X' = [X,X] = \{[a_i,b_i]:a_i,b_i \in X\}$ are studied [ ]. In this paper, we geralized the notion of derived sub-algebra X' by $X' = [X,X] = \{\prod[a_i,b_i]:a_i,b_i \in X\}$.

Example 3.7. Let the X = {0,a,b,c,d} and (*) operation be the following table.

76. *	77. 0	78. a	79. b	80. C	81. D
82. 0	83. 0	84. 0	85. 0	86. 0	87. 0
88. a	89. a	90. 0	91. a	92. 0	93. a
94. b	95. b	96. b	97. 0	98. 0	99. 0
100. c	101. c	102. c	103. c	104. 0	105. c
106. d	107. d	108. d	109. b	110. B	111. 0

It is not difficult to verify that (X,*,0) is a BCIK-algebra. Consider A = {0,a} and B = {0,c}, then A and B are sub-algebra of X. It is easy to check that [A,B] = {0} and [B,A] = {0,a}, therefore [A,B] ≠ [B,A]. Now, X' = {0,a,b} is a sub-algebra of X, but X' is not an ideal of X because d*b = b Є X' and b Є X' but d not belongs to X'. Its initial segments of X are [0,a] = {0,a},[0,b] = {0,b},[0,c] = {0,a,b,c}, [0,d] = {0,b,d}. Hence X' is not an initial segment of X.

Remark 3.8. In Example 3.2. ,considering A = {0,2} and B = {0,1,2,3,4}, we see A and B are two ideals of X and [A,B] = {0,1}, but [A,B] is not a sub-algebra of B.

Theorem 3.9.

(1) X is commutative if and only if X' = {0},

(2) X' is sub-algebra of X.

Corollary 3.10. Let (X,*,0) be a commutative BCIK-algebra. If A and B are two subsets of X, then [A,B] = [B,A] = {0}.

Example 3.11. Let X = {0,a,1} and (*) operation be given by the following table

112. *	113. 0	114. a	115. 1
116. 0	117. 0	118. 0	119. 0
120. a	121. a	122. 0	123. 0
124. 1	125. 1	126. a	127. 0

Then (X,*,0) is a commutative bounded BCIK-algebra with the largest element 1. Then A = {0,1}, B={0,a} are two sub-algebras of X and [A,B] = [B,A] = {0}.

Example 3.12. Let X = {0,1,2,….,n} and * is given

x*y = 0 if x≤y

x otherwise

Then (X,*,0) is a bounded positive implicative BCIK-algebra of order n. X' = {0,1,2,…….,n-1}, therefore X' is of order n-1. Also X'' = {0,1,2,……..,n-3} and $X^{(3)}$ = {0,1,2,…….,n-4},……$X^{(n-1)}$ = {0}.

Theorem 3.12. If Y is a sub-algebra of X, then Y' is a sub-algebra of X'.

Proof. Let x Є Y'. Then there exist a,b Є Y such that x = [a,b], but Y contain X, then a,b Є X and x = [a,b]. Hence x Є X' and so Y' contain X'. However, Y' and X' are two sub-algebras of X. Therefore Y' is a sub-algebra of X'.

Theorem 3.13. Let I be an ideal of X. Then X/I are commutative BCIK-algebras if and only if X' contain I.

Proof. X/I are commutative BCIK-algebra if and only if for all x,y Є X, $C_x \wedge C_y = C_y \wedge C_x$ if and only if for all x,y Є X, $C_{x\wedge y} = C_{x\vee y}$ if and only if for all x, y Є X, [x,y]≈0 if and only if for all x, y Є X, [x,y] *0 Є I and 0*[x,y] Є I if and only if for all x, y Є X, [x,y] Є I if and only if X' contain I.

4. Final Result

Definition 4.1. X is called a solvable BCIK-algebra, if there exists a non-negative real number n such that $X^{(n)}$ = {0}.

Note that if X is a commutative BCIK-algebra, then X is a solvable pseudo commutators BCIK-algebra. Since any plicative BCIK-algebra X is a commutative BCIK-algebra[18], then X' = {0} for any implicative BCIK-algebra X, so any implicative BCIK-algebra X is a solvable BCIK-algebra.

Example 4.2. [18] Let X = {0,a,b,c,d,e,f,1} and (*) operation be given by the following table

5. *	6. 0	7. A	8. b	9. c	10. d	11. e	12. f	13. 1
14. 0	15. 0	16. 0	17. 0	18. 0	19. 0	20. 0	21. 0	22. 0
23. a	24. a	25. 0	26. 0	27. 0	28. A	29. 0	30. 0	31. 0
32. b	33. b	34. A	35. 0	36. 0	37. b	38. a	39. 0	40. 0
41. c	42. c	43. A	44. a	45. 0	46. c	47. a	48. a	49. 0
50. d	51. d	52. D	53. d	54. d	55. D	56. 0	57. 0	58. 0
59. e	60. e	61. D	62. d	63. d	64. a	65. 0	66. 0	67. 0
68. f	69. f	70. E	71. d	72. d	73. b	74. a	75. 0	76. 0
77. 1	78. 1	79. e	80. e	81. d	82. c	83. a	84. a	85. 0

It is difficult to verify that (X,*,0) is a bounded BCIK-algebra of order 8. It is easy to check that X'={0,a} and X'' = {0}. So X is a solvable pseudo commutators BCIK-algebra.

Example 4.3. Let X={x Є R: 0≤x≤1} and (*) operation defined on X by x*y = max{0,x-y}. Then (X,*,0) is a BCIK-algebra. We see that X' = {0}. Therefore X is a solvable pseudo commutators BCIK-algebra. But if we define x*y = 0 if x≤y

x otherwise

Then (X,*,0) is a BCIK-algebra and $X^{(n)}$ = [0,1) for any n≥1, so X is not a solvable pseudo commutators BCIK-algebra .

Let f be a homomorphism from BCIK-algebras (X,*,0) to BCIK-algebra (Y,*',0') and x, y Є X. Then f([x,y]) = f((x∧y)*(y∧x)) = f(x∧y)*'f(y∧x) = f(y*(y*x))*'f(x*(x*y))=(f(y)*'f(y*x))*'(f(x)*'f(x*y)) = '(f(y)*'(f(y)*'f(x)))*'(f(x)*'(f(x)*'f(y)) = (f(y)∧f(x))*'(f(x)∧f(y)) = [f(x),f(y)]. i.e, if [x,y]ϵ X',then f([x,y]) = [f(x),f(y)] ϵ Y'.

Theorem 4.3. Let f be an isomorphism from X to BCIK-algebra Y. X is a solvable pseudo commutators BCIK-algebra if and only if Y is a solvable pseudo commutators BCIK-algebra.

Proof. Since f(x) = Y, therefore f(X') = y',because if yϵ f(X'),then there exists axϵX' such that f(x)-y. But xϵX',then there exist a_i,b_i ϵ X, then there exist a_i,b_i ϵ X such that x = $\Pi[a_i,b_i]$. Consequently f(x) = f($\Pi[a_i,b_i]$) = $\Pi f[a_i,b_i]$ = $\Pi[f(a_i),f(b_i)]$ = y ϵ Y'. So, f(X') ⊆ Y'.

Conversely, let h ϵ Y'. Then there exists h_i,k_i ϵ Y such that h = $\Pi[h_i,k_i]$. However, f is an isomorphism, so there exist a_i,b_i ϵ X such that $h_i = f(a_i)$, $k_i = f(b_i)$. Thus, h = $\Pi[h_i,k_i]$ = $\Pi[f(a_i),f(b_i)]$ = $\Pi f[a_i,b_i]$=f($\Pi[a_i,b_i]$) ϵ f(X'). By induction, we can show that f($X^{(n)}$)= $Y^{(n)}$. Since X is a solvable pseudo commutators BCIK-algebra, there exit n ϵ N such that $X^{(n)}$ = {0}. Therefore, {0} = f({0}] = f($X^{(n)}$) = $Y^{(n)}$; that is, Y is solvable pseudo commutators BCIK-algebra.

Conversely, let Y be a solvable pseudo commutators BCIK-algebra. Since f(x) = Y, then f($X^{(n)}$) = $Y^{(n)}$ = {0}, Therefore f($X^{(n)}$) = {0} = f({0}), so $X^{(n)}$ = {0}; that is X is a solvable pseudo commutators BCIK-algebra.

Theorem 4.4. Sub-algebras and homomorphic images of solvable pseudo commutators BCIK-algebra also are solvable pseudo commutators BCIK-algebra.

Proof. Let Y be a sub-algebra of X. Then for any n ϵ N, $Y^{(n)}$ is sub-algebra of $X^{(n)}$. Since X is a solvable pseudo commutators BCIK-algebra, there exists n such that $X^{(n)}$ = {0} and so $Y^{(n)}$ = {0}; that is, Y is a solvable pseudo commutators BCIK-algebra.

Now, let (X,*,0) and (Y,*',0') be two BCIK-algebra and X be a solvable pseudo commutators BCIK-algebra, f:X→Y be a epimorphism (homomorphism0 from X to Y. Then for some non-negative real number n, we have f($X^{(n)}$) = $Y^{(n)}$. Hence {0} = f({0}] = f($X^{(n)}$) = $Y^{(n)}$. So, f($X^{(n)}$) = $Y^{(n)}$ = {0}; that is, Y is a solvable pseudo commutators BCIK-algebra.

Theorem 4.5. Let I be an ideal of X. If I and X/I are solvable pseudo commutators BCIK-algebra, then X is a solvable pseudo commutators BCIK-algebra.

Proof. Let f be natural homomorphism from X onto X/I. Since X/I is solvable pseudo commutators BCIK-algebra, so for some n, f($X^{(n)}$) = $(X/I)^{(n)}$ = {I}. Hence $X^{(n)}$ is a sub-algebra of ker(f) = I. By Theorem 4.4., $X^{(n)}$ is solvable pseudo commutators BCIK-algebra. Therefore there exists a positive integer k such that $X^{(n+k)} = (X^{(n)})^{(k)}$ = {0}. That is X is solvable pseudo commutators BCIK-algebra.

Lemma 4.6. Let A, B ⊆ X. Then

(1) A' ⋃ B' ⊆ (A ⋃ B)',

(2) (A ⋂ B)' ⊆ A' ⋂ B'.

Proof. (1). Since A and B are two subsets of (A ⋃ B), A' ⊆ (A ⋃ B)' and B' ⊆ (A ⋃ B)'. Therefore, A' ⋃ B' ⊆ (A ⋃ B)'.

Proof. (2). Since (A ⋂ B) ⊆ A and (A ⋂ B) ⊆ B. Then (A ⋂ B)' ⊆ A' and (A ⋂ B)' ⊆ B'. So (A ⋂ B)' ⊆ A' ⋂ B'. In, general, (A ⋃ B)' ⊄ A' ⋃ B' and (A ⋂ B)' ⊄ A' ⋂ B'. For example, in the BCIK-algebra in Example 3.7. consider A = {0,a} and B = {0,c}. Then A' = {0}, B' = {0} and (A ⋃ B)' = {0,a} ⊄ A' ⋃ B' = {0}.

Moreover in Example 3.8., consider A= {0,1} and B = {a}, then A' = B' = {0} and A' ⋂ B'
= {0} ⊄ (A ⋂ B)' = φ .

Theorem 4.7. The intersection of any two solvable sub-algebra of X, is solvable.

Proof. Let $X_1$and X_2be the two solvable sub-algebras of X. Since $(X_1 \bigcap X_2)^{(n)} \subseteq X_1^{(n)}$ = {0}. Then $(X_1 \bigcap X_2)^{(n)}$ = {0}, that is $(X_1 \bigcap X_2)$ is a solvable pseudo commutators BCIK-algebra.

The above result can be generalized such that the intersection of any arbitrary family of sub-algebras of a solvable pseudo commutators BCIK-algebra, is again a solvable pseudo commutators BCIK-algebra. However, in general, the union of two solvable pseudo commutators BCIK sub-algebra of a BCIK-algebra may not be a solvable pseudo commutators BCIK-algebra. For example, consider the BCIK-algebra (N,*,0) together with x*y = max{0,x-y}, for all x, y ϵ N. Let X_1 be the sub-algebra of multiples of with the operation * and X_2 be the sub-algebra of multiples of three with operation *. Observe that $X_1 \bigcup X_2$ is not a sub- algebra, because it is not closed under *.

We have only one BCIK-algebra of order one, that is , X={0}, for this algebra X' = {0}. Also there is a unique BCIK-algebra of order two, that is X={0,1} with the following operation[6] for this algebra X' = {0}. Up to isomorphism, there exist only three BCIK-algebra of order 3

86. *	87. 0	88. 1
89. 0	90. 0	91. 0
92. 1	93. 1	94. 0

And fourteen BCIK-algebra of order 4 and eight BCIK-algebra of order 5 all being solvable pseudo commutators BCIK-algebra, so all BCIK-algebra with order less than 6 are solvable pseudo commutators BCIK-algebra.

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