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Observations on $2y^2 + xy = z^2$

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Abstract

Formulae for generating sequences of integer solutions based on the given solution to the ternary homogeneous quadratic Diophantine equation given by $2y^2 + xy = z^2$ are exhibited.

Keywords: homogeneous equation, ternary quadratic, generation of solutions

Introduction

The subject of diophantine equations is one of the significant areas in number theory and occupies a remarkable position in history due to its unquestioned historical importance .The purpose of any diophantine equation is to solve for all the unknowns in the problem. It is quite obvious that diophantine equations are rich in variety and there are methods available to obtain solutions either in real integers or in Gaussian integers.

A natural question that arises now is ,whether a general formula for generating sequence of solutions based on the given solution can be obtained? While searching for problems on quadratic diophantine equations, the authors came across the book [1] entitled "CONSTRUCTION OF GENERATION FORMULA FOR SEQUENCE OF INTEGER SOLUTIONS TO SPECIAL HOMOGENEOUS CONES". The results presented in the above book motivated us for obtaining sequences of integer solutions based on the given solution to the ternary homogeneous quadratic diophantine equation given by $2y^2 + xy = z^2$.

Method of analysis:

The homogeneous quadratic equation with three unknowns under consideration is

$$2y^2 + xy = z^2 \tag{1}$$

After performing a few algebra ,(1) is satisfied by the following triples:

$(x, y, z) = (8r^2 - s^2, -4r^2, 2rs), (8r^2 - 4k^2, 2k^2, 4rk), (k^2 - 2, 1, k), (4k^2 - 2, -2k^2, 2k),$
$(2k^2 - 4, 2, 2k), (2k^2 - 4, -k^2, 2k), (2k^2 - 1, -k^2, k), ((k^2 + 2k - 1)s, s, (k+1)s), (14k, k, 4k),$
$(7k, -4k, 2k), (a^2 - 2ab - b^2, b^2, ab - b^2), (-a^2 + 2ab + b^2, a^2, ab + a^2)$

It is observed that ,if (x_0, y_0, z_0) is any given solution to (1), then ,the triple

$(x_0, -x_0 - 3y_0 - 2z_0, x_0 + 4y_0 + 3z_0)$ also satisfies (1).

Now, formulae for generating sequences of integer solutions based on the given solution to the ternary homogeneous quadratic Diophantine equation given by $2y^2 + xy = z^2$ are exhibited

Formula 1:

Let (x_0, y_0, z_0) be any solution to (1).

Let (x_1, y_1, z_1) be the second solution to (1),where

$$x_1 = x_0 + h, y_1 = y_0, z_1 = h - z_0 \quad (2)$$

in which h is an unknown to be determined.

Substitution of (2) in (1) gives

$$h = y_0 + 2z_0 \quad (3)$$

Using (3) in (2) ,the second solution (x_1, y_1, z_1) to (1) is expressed in the matrix form as

$$(x_1, y_1, z_1)^t = M(x_0, y_0, z_0)^t$$

where t is the transpose and

$$M = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

The repetition of the above process leads to the general solution (x_n, y_n, z_n) to (1) written in the matrix form as

$$(x_n, y_n, z_n)^t = M^n (x_0, y_0, z_0)^t$$

where

$$M^n = \begin{bmatrix} 1 & n^2 & 2n \\ 0 & 1 & 0 \\ 0 & n & 1 \end{bmatrix}$$

Formula 2:

Let

$$x_1 = x_0 - h, y_1 = y_0 + h, z_1 = z_0 \qquad (4)$$

be the second solution of (1). Substituting (4) in (1) and performing a few calculations ,

it is seen that

$$h = -x_0 - 3y_0$$

and thus

$$x_1 = 2x_0 + 3y_0, y_1 = -x_0 - 2y_0$$

which is written in the matrix form as

$$(x_1, y_1)^t = M(x_0, y_0)^t$$

where

$$M = -\begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$$

Proceeding in a similar manner, the general formula for obtaining a sequence of non-zero distinct integer solutions based on the given solution to (1) is represented by

$$(x_n, y_n)^t = M^n (x_0, y_0)^t \;, z_n = z_0$$

where

$$M^n = \frac{1}{2}\begin{bmatrix} 3-(-1)^n & 3-3(-1)^n \\ -1+(-1)^n & -1+3(-1)^n \end{bmatrix}$$

Formula 3:

Let

$$x_1 = x_0 + h, y_1 = y_0 + h, z_1 = 2h - z_0 \qquad (5)$$

be the second solution of (1). Substituting (5) in (1) and performing a few calculations , it is seen that

$$h = x_0 + 5y_0 + 4z_0 \tag{6}$$

Using (6) in (5) ,the second solution (x_1, y_1, z_1) to (1) is expressed in the matrix form as

$$(x_1, y_1, z_1)^t = M(x_0, y_0, z_0)^t$$

where t is the transpose and

$$M = \begin{bmatrix} 2 & 5 & 4 \\ 1 & 6 & 4 \\ 2 & 10 & 7 \end{bmatrix}$$

The repetition of the above process leads to the general solution (x_n, y_n, z_n) to (1) written in the matrix form as

$$(x_n, y_n, z_n)^t = M^n (x_0, y_0, z_0)^t$$

where

$$M^n = \begin{bmatrix} \frac{Y_{n-1}+5}{6} & \frac{5Y_{n-1}-5}{6} & X_{n-1} \\ \frac{Y_{n-1}-1}{6} & \frac{5Y_{n-1}+1}{6} & X_{n-1} \\ \frac{X_{n-1}}{2} & \frac{5X_{n-1}}{2} & Y_{n-1} \end{bmatrix}$$

in which

$$Y_{n-1} = \frac{(7+4\sqrt{3})^n + (7-4\sqrt{3})^n}{2},$$

$$X_{n-1} = \frac{(7+4\sqrt{3})^n - (7-4\sqrt{3})^n}{2\sqrt{3}}, n = 1,2,3,...$$

To conclude ,the readers of this paper may search for other choices of quadratic diophantine equations with three or more unknowns to formulate the generation formula for sequence of integer solutions based on the known solution.

Reference

[1]. M.A.Gopalan,S.Vidhyalakshmi,S.Aarthy Thangam, J.Srilekha, “CONSTRUCTION OF GENERATION FORMULA FOR SEQUENCE OF INTEGER SOLUTIONS TO SPECIAL HOMOGENEOUS CONES”,Mahi Publication,Ahmedabad,2019