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On Homogeneous Cubic Equation with Five Unknowns $x^3 + y^3 = 13(z + w)P^2$

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ABSTRACT :

The homogeneous cubic equation with five unknowns given by $x^3 + y^3 = 13(z + w)P^2$ is considered for obtaining its non-zero distinct integer solutions through employing linear transformations.

Keywords: homogeneous cubic, cubic with five unknowns, integer solutions

Introduction:

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular, cubic equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity [1-3]. For illustration, one may refer [4-13] for homogeneous and non-homogeneous cubic equations with three, four and five unknowns. This paper concerns with the problem of determining non-trivial integral solution of the non-homogeneous cubic equation with five unknowns given by $x^3 + y^3 = 13(z + w)P^2$.

Method of analysis:

The homogeneous cubic equation with five unknowns to be solved is

$$x^3 + y^3 = 13(z+w)p^2 \tag{1}$$

Substitution of the linear transformations

$$x = 13(u+v),\, y = 13(u-v),\, z = u+d,\, w = u-d,\, p = 13q \tag{2}$$

in (1) leads to

$$u^2 + 3v^2 = q^2 \tag{3}$$

Solving (3) through different ways, the corresponding values of u,v,q are obtained.

Substituting these values in (2), the respective solutions to (1) are determined.

The above process is illustrated below:

Way 1:

It is observed that (3) is satisfied by

$$v = 2rs,\, u = 3r^2 - s^2,\, q = 3r^2 + s^2 \tag{4}$$

In view of (2), the corresponding integer solutions to (1) are given by

$$\begin{aligned} x &= 13(3r^2 - s^2 + 2rs) \\ y &= 13(3r^2 - s^2 - 2rs) \\ z &= 3r^2 - s^2 + d \\ w &= 3r^2 - s^2 - d \\ p &= 13(3r^2 + s^2) \end{aligned}$$

Way 2:

Represent (3) as the system of double equations as in Table:1 below :

Table: 1 System of double equations

System	I	II	III	IV
$q+u$	$3v^2$	v^2	1	3
$q-u$	1	3	$3v^2$	v^2

Solving each of the above system of equations in Table:1, the values of q, u, v are obtained.

Using these values in (2), the corresponding integer solutions to (1) are found .For brevity, the respective solutions to (1) are exhibited as below:

Solutions from system I:

$$x = 13(6k^2 + 8k + 2)$$
$$y = 13(6k^2 + 4k)$$
$$z = 6k^2 + 6k + 1 + d$$
$$w = 6k^2 + 6k + 1 - d$$
$$p = 13(6k^2 + 6k + 2)$$

Solutions from system II:

$$x = 13(2k^2 + 4k)$$
$$y = 13(2k^2 - 2)$$
$$z = 2k^2 + 2k - 1 + d$$
$$w = 2k^2 + 2k - 1 - d$$
$$p = 13(2k^2 + 2k + 2)$$

Solutions from system III:

$$x = 13(-6k^2 - 4k)$$
$$y = 13(-6k^2 - 8k - 2)$$
$$z = -6k^2 - 6k - 1 + d$$
$$w = -6k^2 - 6k - 1 - d$$
$$p = 13(6k^2 + 6k + 2)$$

Solutions from system IV:

$$x = 13(-2k^2 + 2)$$
$$y = 13(-2k^2 - 4k)$$
$$z = -2k^2 - 2k + 1 + d$$
$$w = -2k^2 - 2k + 1 - d$$
$$p = 13(2k^2 + 2k + 2)$$

Way 3:

Write (3) as

$$u^2 + 3v^2 = q^2 * 1 \tag{5}$$

Assume

$$q = a^2 + 3b^2 \tag{6}$$

and 1 on the R.H.S. of (5) is written as

$$1 = \frac{(1 + i\sqrt{3})(1 - i\sqrt{3})}{4} \tag{7}$$

Substituting (6) & (7) in (5) and employing the method of factorization ,define

$$u + i\sqrt{3}v = \frac{(1 + i\sqrt{3})(a + i\sqrt{3}b)^2}{2} \tag{8}$$

Equating the real and imaginary parts in (8) ,the values of u and v are obtained.

In view of (2), the corresponding values of x ,y ,z ,w and p are determined and they are as below:

$$\left.\begin{aligned} x &= 13(a^2 - 3b^2 - 2ab) \\ y &= -52ab \\ z &= \frac{(a^2 - 3b^2)}{2} - 3ab + d \\ w &= \frac{(a^2 - 3b^2)}{2} - 3ab - d \\ p &= 13(a^2 + 3b^2) \end{aligned}\right\} \tag{9}$$

As our interest is on finding integer solutions observe that both a and b should be of the same parity in (9).

Note : 1 The integer 1 on the R.H.S. of (5) may also be represented as follows:

$$1=\frac{(1+i4\sqrt{3})(1-i4\sqrt{3})}{49}$$

$$1=\frac{(3r^2-s^2+i\sqrt{3}\,2rs)(3r^2-s^2-i\sqrt{3}\,2rs)}{(3r^2+s^2)^2}$$

Following the above procedure two more sets of integer solutions to (1) are obtained.

Way 4:

Rewrite (3) as

$$q^2-3v^2=u^2*1 \tag{10}$$

Assume

$$u=a^2-3b^2 \tag{11}$$

and take 1 on the R.H.S. of (10) as

$$1=\left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right) \tag{12}$$

Substituting (11) & (12) in (10) and applying the method of factorization, define

$$q+\sqrt{3}v=(a+\sqrt{3}b)^2(2+\sqrt{3}) \tag{13}$$

Equating the rational and irrational parts, we have

$$v=a^2+3b^2+4ab$$
$$q=2a^2+6b^2+6ab$$

In view of (2), the integer solutions to (1) are given by

$$x=13(2a^2+4ab)$$
$$y=13(-6b^2-4ab)$$
$$z=a^2-3b^2+d$$
$$w=a^2-3b^2-d$$
$$p=13(2a^2+6b^2+6ab)$$

Note: 2

The integer 1 on the R.H.S. of (10) may also be represented as follows:

$$1=(7+4\sqrt{3})\,(7-4\sqrt{3})$$

$$1=\frac{(3r^2+s^2+\sqrt{3}\,2rs)(3r^2+s^2-\sqrt{3}\,2rs)}{(3r^2-s^2)^2}$$

Following the above procedure two more sets of integer solutions to (1) are obtained.

Way 5:

Introduction of the linear transformations

$$u=X+3T,\ v=X-T,\ q=2R \tag{14}$$

in (3) leads to

$$X^2+3T^2=R^2$$

which is satisfied by

$$T=2ab,\ X=3a^2-b^2,\ R=3a^2+b^2 \tag{15}$$

Using (15) in (14) and employing (2),the integer solutions to (1) are found to be

$$\begin{aligned}
x&=13(6a^2-2b^2+4ab)\\
y&=104ab\\
z&=3a^2-b^2+6ab+d\\
w&=3a^2-b^2+6ab-d\\
p&=13(6a^2+2b^2)
\end{aligned}$$

Note: 3

Instead of (14), suppose we have the transformations

$$u=X-3T,\ v=X+T,\ q=2R$$

In this case, the corresponding integer solutions to (1) are found to be

$$x = 13(6a^2 - 2b^2 - 4ab)$$
$$y = -104ab$$
$$z = 3a^2 - b^2 - 6ab + d$$
$$w = 3a^2 - b^2 - 6ab - d$$
$$p = 13(6a^2 + 2b^2)$$

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