International Journal of Research Publication and Reviews

Journal homepage: www.ijrpr.com ISSN 2582-7421

On Para-Kenmotsu Manifolds

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A B S T R A C T

The object of the present paper is to study the condition of semi-symmetric type on para-Kenmotsu manifold.To show that a semi-symmetric para-Kenmotsu manifold is an η-Einstein manifold satisfies the condition $\tilde{S}.\tilde{R} = 0$ and on para-Kenmotsu manifold satisfies curvature properties $\tilde{R}.\tilde{R} = 0$ and $\tilde{R}.\tilde{S} = 0$ is an Einstein manifold .
AMS Mathematics Subject Classification (2010): 53C15, 53C25.

Keywords: para-Kenmotsu manifold, η- Einstein manifold , Einstein manifold

1 Introduction

The notion of an almost para-contact manifold was introduced by I. Sato [7]. Since the publication of [10], paracontact metric manifolds have been studied by many authors in recent years. In this paper, consider the para-Kenmotsu manifolds. Note that the para-Kenmotsu structure was introduced by J. We lyczko in [13] for 3-dimensional normal almost paracontact metric structures. A similar notion called P-Kenmotsu structure appears in the paper of B. B. Sinha and K. L. Sai Prasad [12] .The importance of para-Kenmotsu geometry has been pointed out especially in the last years by several papers highlighting the exchanges with the theory of para-K$\ddot{a}$hler manifolds and its role in semi-Riemannian geometry and mathematical physics [3, 4, 5, 8, 9].

The present paper is organized as follows: After preliminaries on para-Kenmotsu manifold in section 2. We study in section 3, the semi-symmetric type condition on para-Kenmotsu manifold is an η-Einstein manifold. In section 4. a Ricci semi-symmetric condition on para-Kenmotsu manifold is an η-Einstein manifold. Further, In the section 5, Semi-symmetric type condition on para-Kenmotsu manifold is an η-Einstein manifold

2 Preliminaries

An n-dimensional differentiable manifold M^n is said to have almost para-contact structure (ϕ, ξ, η), where ϕ is a tensor field of type $(1,1)$, ξ is a vector field known as characteristic vector field and η is a 1-form satisfying the following relations

$$\phi^2(X) = X - \eta(X)\xi, \tag{2.1}$$

$$\eta(\phi X) = 0, \quad (2.2)$$

$$\phi(\xi) = 0, \quad (2.3)$$

and

$$\eta(\xi) = 1. \quad (2.4)$$

A differentiable manifold with almost para-contact structure (ϕ, ξ, η) is called an almost para-contact manifold. Further, if the manifold M^n has a semi-Riemannian metric g satisfying

$$\eta(X) = g(X, \xi) \quad (2.5)$$

and

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y). \quad (2.6)$$

Then the structure (ϕ, ξ, η, g) satisfying conditions (2.1) to (2.6) is called an almost para-contact Riemannian structure and the manifold M^n with such a structure is called an almost para-contact Riemannian manifold [1, 7].

Now we briefly present an account of an analogue of the Kenmotsu manifold in para-contact geometry which will be called para-Kenmotsu.

Definition 2.1 *The almost paracontact metric structure (ϕ, ξ, η, g) is para-Kenmotsu should this relation hold[2, 11], if the Levi-Civita connection ∇ of g satisfies $(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$, for any $X, Y \in \mathfrak{X}(M)$.*

On a para-Kenmotsu manifold [2, 8], by the following relations hold:

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \quad (2.7)$$

$$\nabla_X \xi = X - \eta(X)\xi, \quad (2.8)$$

$$(\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y), \quad (2.9)$$

$$\eta(R(X, Y, Z)) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \quad (2.10)$$

$$R(X, Y, \xi) = \eta(X)Y - \eta(Y)X, \quad (2.11)$$

$$R(X, \xi, Y) = -R(\xi, X, Y) = g(X, Y)\xi - \eta(Y)X, \quad (2.12)$$

$$S(\phi X, \phi Y) = -(n-1)g(\phi X, \phi Y), \quad (2.13)$$

$$S(X, \xi) = -(n-1)\eta(X), \quad (2.14)$$

$$Q\xi = -(n-1)\xi, \quad (2.15)$$

$$r = -n(n-1), \quad (2.16)$$

for any vector fields X, Y, Z , where Q is the Ricci operator that is $g(QX, Y) = S(X, Y)$, S is the Ricci tensor and r is the scalar curvature.

In A. M. Blaga [2], has given an example on the para-Kenmotsu manifold.

Example 2.1 *[2] We consider the three dimensional manifold* $M^3 = \{(x,y,z) \in \mathbb{R}^3, z \neq 0\}$, *where* (x,y,z) *are the standard co-ordinates in* $\mathbb{R}^3$. *The vector fields*

$$e_1 := \frac{\partial}{\partial x}, \quad e_2 := \frac{\partial}{\partial y}, \quad e_3 := -\frac{\partial}{\partial z}$$

are linearly independent at each point of the manifold.

Define

$$\phi := \frac{\partial}{\partial y} \otimes dx + \frac{\partial}{\partial x} \otimes dy, \;\; \xi := -\frac{\partial}{\partial z}, \;\; \eta := -dz,$$

$$g := dx \otimes dx - dy \otimes dy + dz \otimes dz.$$

Then it follows that

$$\phi e_1 = e_2, \quad \phi e_2 = e_1, \quad \phi e_3 = 0,$$

$$\eta(e_1) = 0, \;\; \eta(e_2) = 0, \;\; \eta(e_3) = 1.$$

Let ∇ be the Levi-Civita connetion with respect to metric g. Then, we have

$$[e_1, e_2] = 0, \;\; [e_2, e_3] = 0, \;\; [e_3, e_1] = 0.$$

The Riemannian connection ∇ of the metric g is deduced from Koszul's formula

$$\begin{aligned} 2g(\nabla_X Y, Z) \quad &= X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\ &-g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]). \end{aligned}$$

Then Koszul's formula yields

$$\nabla_{e_1} e_1 = -e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = e_1,$$

$$\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = e_3, \quad \nabla_{e_2} e_3 = e_2,$$

$$\nabla_{e_3} e_1 = e_1, \quad \nabla_{e_3} e_2 = e_2, \quad \nabla_{e_3} e_3 = 0.$$

These results show that the manifold satisfies

$$\nabla_X \xi = X - \eta(X)\xi,$$

for $\xi = e_3$. Hence, the manifold under consideration is para-Kenmotsu manifold of dimension three.

A para-Kenmotsu manifold is said to be an η-Einstein manifold if its Ricci tensor S is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) \tag{2.17}$$

for the vector fields X, Y, where a and b are functions on M^n.

3 Semi-Symmetric on Para-Kenmotsu Manifold

Definition 3.1 *A para-Kenmotsu manifold is said to be semi-symmetric [9] if the condition*

$$R(X,Y) \cdot R = 0, \tag{3.1}$$

holds for all vector fields X, Y.

Theorem 3.1 *A semi-symmetric condition on para-Kenmotsu manifold is an Einstein manifold.*

Proof: Consider

$$(R(X,Y)\cdot R)(Z,W,U)=0 \tag{3.2}$$

for all vector fields X, Y, Z, W and U. Then, in particular for $X=\xi$, we have

$$(R(\xi,X)\cdot R)(Y,Z,W)=0,$$

which, implies that

$$\begin{aligned} 0 &= R(\xi,X,R(Y,Z,W))-R(R(\xi,X,Y),Z,W) \\ &-R(Y,R(\xi,X,Z),W)-R(Y,Z,R(\xi,X,W). \end{aligned}$$

Now, using (2.10), (2.12) in the above equation, we find that

$$\begin{aligned} 'R(Y,Z,W,X)\xi &= [g(Y,W)\eta(Z)-g(Z,W)\eta(Y)]X \\ &-\eta(Y)R(X,Z,W)+g(X,Y)R(\xi,Z,W)-\eta(Z)R(Y,X,W) \\ &+g(X,Z)R(Y,\xi,W)-\eta(W)R(Y,Z,X)+g(X,W)R(Y,Z,\xi). \end{aligned}$$

Again, using the equations (2.10), (2.11) and (2.12), in the above equation and taking inner product with ξ, we get

$$'R(Y,Z,W,X)=g(X,Z)g(Y,W)-g(X,Y)g(Z,W).$$

Now, contracting the above equation with respect to X and Y, we obtain

$$S(W,Z)=(1-n)g(W,Z).$$

This proves the theorem.

4 Ricci Semi-Symmetric on Para-Kenmotsu Manifold

Definition 4.1 *A para-Kenmotsu manifold is said to be Ricci semi-symmetric [6] if the condition*

$$R(X,Y)\cdot S=0, \tag{4.1}$$

holds for all vector fields X, Y.

Theorem 4.1 *A Ricci semi-symmetric condition on para-Kenmotsu manifold is an Einstein manifold.*

Proof: Consider

$$(R(X,Y)\cdot S)(Z,W)=0,$$

which gives

$$S(R(\xi,Y,Z),W)+S(Z,R(\xi,Y,W))=0.$$

Using equations (2.12), (2.14) in above equation, we get

$$\begin{aligned} &S(Y,W)\eta(Z)+(n-1)g(Y,Z)\eta(W) \\ &+S(Y,Z)\eta(W)+(n-1)g(Y,W)\eta(Z)=0. \end{aligned}$$

Putting $Z=\xi$ in the above equation and using (2.4), (2.5) and (2.14), we get

$$S(Y,W)=-(n-1)g(Y,W)$$

This proves the theorem.

5 Semi-Symmetric Type on Para-Kenmotsu Manifold

Definition 5.1 *A para-Kenmotsu manifold is said to be semi-symmetric type if the condition*

$$S(X,Y)\cdot R=0, \tag{5.1}$$

holds for all vector fields X,Y.

Theorem 5.1 *A semi-symmetric type condition on para-Kenmotsu manifold is an η-Einstein manifold.*

Proof: Consider

$$S(\xi,X)\cdot R=0,$$

which gives

$$\begin{aligned}&S(X,R(Y,Z,W))\xi-S(\xi,R(Y,Z,W))X+S(Y,X)R(\xi,Z,W)\\&-S(Y,\xi)R(X,Z,W)+S(X,Z)R(Y,\xi,W)-S(Z,\xi)R(Y,X,W)\\&+S(W,X)R(Y,Z,\xi)-S(W,\xi)R(Y,Z,X)=0.\end{aligned}$$

Taking inner product with ξ and using equations (2.5), (2.10), (2.12), (2.14) in above equation, we get

$$\begin{aligned}&S(X,R(Y,Z,W))+2(n-1)\eta(X)\eta(Z)g(Y,W)-2(n-1)\eta(X)\eta(Y)g(Z,W)\\&+\eta(W)\eta(Z)S(Y,X)-S(Y,X)g(Z,W)+S(Z,X)g(Y,W)-\eta(W)\eta(Y)S(Z,X)\\&+(n-1)\eta(W)\eta(Z)g(Y,X)-(n-1)\eta(W)\eta(Y)g(Z,X)=0.\end{aligned}$$

Putting $W=\xi$ in the above equation and using (2.11), we get

$$\eta(Y)S(Z,X)-\eta(Z)S(Y,X)+(n-1)\eta(Z)g(Y,X)-(n-1)\eta(Y)g(Z,X)=0.$$

Again putting $Z=\xi$ in the above equation and using (2.4), (2.5), (2.14), we get

$$S(Y,X)=(n-1)g(Y,X)-2(n-1)\eta(X)\eta(Y).$$

This proves the theorem.

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